Demonstration of universal quantum computation on decoherence-free-subspaces with ions in thermal motion

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Abstract

We propose a procedure to demonstrate fault-tolerant universal quantum gates on decoherence-free subspaces with ions trapped in a linear Paul trap, using interactions between only two ions at a time. The ions are not required to be in their motional ground state and are individually or pairwise addressed by laser fields. This implementation is within reach of present technology. We also derive the general form of the Hamiltonian involving only two-qubit interactions that generates fault-tolerant universal quantum computation even if the two pairs of qubits used to encode the two logical qubits are farther away than the noise correlation length.

Introduction–In quantum information processing tasks decoherence can be overcome either by an active approach or by a passive one. The former consists, in analogy with classical computation, of encoding information in a redundant fashion by means of the so-called error-correcting codes [1]. In this approach information is encoded in subspaces of the total Hilbert space in such a way that "errors" induced by the interaction with the environment can be detected and corrected without gaining information about the actual state of the system prior to corruption. The passive approach, on the other hand, is an error prevention scheme, in which logical qubits are encoded within subspaces that do not loose their coherence for reasons of symmetry. The conditions for the existence and the formal development of such decoherence-free subspaces (DFS's) can be found elsewhere [2]. On the experimental side, demonstration of immunity of a DFS of two photons to collective noise was accomplished in [3] and realizations of DFS's for nuclear magnetic resonance (NMR) systems were carried out in [4]. Nevertheless, here we concentrate on trapped-ions, where substantial progress in that direction has been made [5, 6]. These experiments demonstrate that for trapped ions global dephasing is the major source of qubit decoherence; that is, all qubits experience the same external perturbation affecting only the qubit phases. Therefore, we consider only dephasing noise throughout the rest of the work and refer to phase-decoherence-free simply as decoherence-free. A universal non-local gate between two encoded logical qubits is yet to be demonstrated though.

The essential idea behind DFS's can be summarized in the following statement: any coherent superposition of states with the same amount of excitations is immune against collective dephasing due to fluctuations (of the magnetic field, the laser frequency, etc.). In fact, in the presence of fluctuating fields, a general qubit-state $|\Psi\rangle \equiv a|0\rangle + b|1\rangle$ transforms as $|\Psi\rangle \to a|0\rangle + be^{i\zeta}|1\rangle$, which leads to the loss of coherence of the state for ζ is a random fluctuating phase. Using the subspace of the total Hilbert space of two physical qubits spanned by the basis $B_L \equiv \{|0_L\rangle \equiv |01\rangle; |1_L\rangle \equiv |10\rangle\}$ (where the subindex "L" stands for logical) to encode information, we protect it from the dephasing provoked by global (spatially uniform) fluctuations; that is, the logical state $|\Psi_L\rangle \equiv a|0_L\rangle + b|1_L\rangle = a|0\rangle|1\rangle + b|1\rangle|0\rangle$ evolves under the previous transformation as $|\Psi_L\rangle \to a|0\rangle e^{i\zeta}|1\rangle + be^{i\zeta}|1\rangle|0\rangle = e^{i\zeta}(a|0_L\rangle + b|1_L\rangle)$ and is thus invariant under such evolution up to an irrelevant global phase factor. We call this protected subspace $\mathbb{V}_{DFS_2}$. Four physical qubits are in turn needed to realize two logical ones. The direct product $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$, which is spanned by the basis $B_L \otimes B_L$, yields a DFS in this case. However one should note that this is not the total protected subspace supported by all four qubits, as the states $|0011\rangle$ and $|1100\rangle$, which are outside $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$, are also protected against global dephasing, for they have the same amount of excitations as the states in $B_L \otimes B_L$. Thus the total protected subspace, which we call $\mathbb{V}_{DFS_4}$, is that spanned by the states in $B_L \otimes B_L$ plus the states $|0011\rangle$ and $|1100\rangle$.

In this paper, we propose a procedure to demonstrate fault-tolerant universal quantum gates on trapped-ions involving interactions between only two ions at a time, which can be immediately implemented with present technology. The proposal is based on an effective Hamiltonian which involves only two-qubit interactions and does not depend on the ion motion, so that the ions are not required to be in their motional ground state provided that they always remain in the Lamb-Dicke regime. It uses the encoding recoupling scheme originally presented in [7] for NMR systems, where a sequence of transformations momentarily take the composite state of the two pairs of qubits out of $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$ but not out of $\mathbb{V}_{DFS_4}$. In this way, protection against global dephasing is achieved throughout the procedure, thus automatically guaranteeing that the scheme be fault-tolerant. One should note that leakage out of $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$ into $\mathbb{V}_{DFS_4}$ is allowed as long as all four qubits experience the same phase fluctuations, which is a realistic assumption if they are not far apart, as compared to the typical coherence length of the fluctuations. Therefore we also derive, for any system of qubits subject to dephasing, the general form of the Hamiltonian involving only two-qubit interactions that generates fault-tolerant universal quantum computation even if the two pairs of qubits used to encode the two logical qubits are far apart, as long as the two physical qubits in each pair do experience the same fluctuations. That is, a Hamiltonian that maps $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$ into itself.

The logical SU(2) Lie Algebra and the two-physical-qubit Hamiltonian–First we want to find a complete set of orthogonal operators that map $\mathbb{V}_{DFS_2}$ into itself (we do not want to exit $\mathbb{V}_{DFS_2}$ in the middle of a computation) and whose action on B_L be equivalent to that of the usual Pauli operators on the computational states of one physical qubit. We define then the logical identity and Pauli operators of the i^{th} logical qubit, $\hat{\sigma}^0_{L_i} \equiv \hat{I}_{L_i}$, $\hat{\sigma}^1_{L_i} \equiv \hat{\sigma}^x_{L_i}$, $\hat{\sigma}^2_{L_i} \equiv \hat{\sigma}^y_{L_i}$ and $\hat{\sigma}^3_{L_i} \equiv \hat{\sigma}^z_{L_i}$, as:

$$\begin{aligned}
\hat{\sigma}^0_{L_i} &\equiv \alpha_i \hat{\sigma}^0_i \otimes \hat{\sigma}^0_{i+1} - (1-\alpha_i)\hat{\sigma}^3_i \otimes \hat{\sigma}^3_{i+1}, \\
\hat{\sigma}^1_{L_i} &\equiv \beta_i \hat{\sigma}^1_i \otimes \hat{\sigma}^1_{i+1} + (1-\beta_i)\hat{\sigma}^2_i \otimes \hat{\sigma}^2_{i+1}, \\
\hat{\sigma}^2_{L_i} &\equiv \gamma_i \hat{\sigma}^2_i \otimes \hat{\sigma}^1_{i+1} - (1-\gamma_i)\hat{\sigma}^1_i \otimes \hat{\sigma}^2_{i+1}, \\
\hat{\sigma}^3_{L_i} &\equiv \delta_i \hat{\sigma}^3_i \otimes \hat{\sigma}^0_{i+1} - (1-\delta_i)\hat{\sigma}^0_i \otimes \hat{\sigma}^3_{i+1};
\end{aligned} \tag{1}$$

with $\alpha_i, \beta_i, \gamma_i$ and δ_i any real numbers such that $0 \leq \alpha_i, \beta_i, \gamma_i$ and $\delta_i \leq 1$ and where the subindexes "i" and "$i+1$" on the physical Pauli operators remind us of which qubit's Hilbert space the operator belongs to. These operators map $\mathbb{V}_{DFS_2}$ onto itself and have the desired action on B_L. It is also easy to see that Eq. (1) is the most general way to construct them from the operators that act on the physical qubits. They form a set of four orthonormal operators: $Tr[\hat{\sigma}^j_{L_i}\hat{\sigma}^k_{L_i}] = \delta_{jk}$, with j and $k = 0, 1, 2$ or 3, and therefore a complete orthonormal basis of the subspace of the complex operators acting on $\mathbb{V}_{DFS_2}$ ($Dim[\mathbb{V}_{DFS_2}] = 2$). They also satisfy, in this same space, the desired SU(2) usual commutation relations: $[\hat{\sigma}^j_{L_i}, \hat{\sigma}^k_{L_i}] = 2i\epsilon_{jkl}\hat{\sigma}^l_{L_i}$, for j, k and $l = 1, 2$ or 3; and $[\hat{\sigma}^0_{L_i}, \hat{\sigma}^j_{L_i}] = 0$, for $j = 0, 1, 2$ or 3. We see thus that the logical Pauli operators defined in (1) are a representation of the Lie algebra SU(2) on $\mathbb{V}_{DFS_2}$.

The situation is now completely equivalent to that of a single qubit, with the logical states in B_L and logical operators in (1) playing the role of the physical ones, the important thing to keep in mind though is that these logical operators allow us to operate on the logical states in the same way as their physical counterparts without ever exiting $\mathbb{V}_{DFS_2}$, that is, they automatically provide us with a set of fault-tolerant operators on the DFS. With this at hand we can now write down the Hamiltonian that generates the most general unitary operation on the i^{th} logical qubit, it reads (in adimensional variables):

$$\hat{H}_{L_i} \equiv B^0_i \hat{\sigma}^0_{L_i} + B^1_i \hat{\sigma}^1_{L_i} + B^2_i \hat{\sigma}^2_{L_i} + B^3_i \hat{\sigma}^3_{L_i}; \tag{2}$$

with B^0_i, B^1_i, B^2_i and B^3_i any real numbers that play the role of a "logical magnetic fields."

We are now given four physical qubits that compose the i^{th} and j^{th} logical qubits and want to find a two-physical-qubit-interaction Hamiltonian, $\hat{H}_{L_iL_j}$, mapping $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$ into itself, which generates a universal non-local logical gate. The general form of this Hamiltonian is $\hat{H}_{L_iL_j} \equiv \hat{H}_{L_i} + \hat{H}_{L_j} + \hat{H}_{int_{L_iL_j}}$, where the interaction term is necessarily composed of combinations of products of logical Pauli operators of both logical qubits. The remarkable observation to notice is that $\hat{\sigma}^3_{L_i}$ and $\hat{\sigma}^3_{L_j}$ are the only logical operators that do not involve products of two non-trivial (not the identity) physical operators; any product of two logical Pauli operators of both logical qubits other than $\hat{\sigma}^3_{L_i} \otimes \hat{\sigma}^3_{L_j}$ will contain more than two physical operators. We see therefore that there exists only one Hamiltonian generating fault-tolerant universal quantum computation on the DFS and at the same time preserving the composite state of both logical qubits inside their corresponding encoded subspace $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$, that requires only two-body interactions. It is given by:

$$\hat{H}_{L_iL_j} \equiv \hat{H}_{L_i} + \hat{H}_{L_j} + J_{L_iL_j}\hat{\sigma}^3_{L_i} \otimes \hat{\sigma}^3_{L_j}, \tag{3}$$

where $J_{L_iL_j}$ is a coupling constant.

Implementation on trapped-ions–Let us now consider N ions of internal transition frequency ω_0 trapped in a linear Paul trap of axial frequency ν where each ion can individually be addressed by a laser field of frequency ω. The easiest logical Pauli operator to implement is $\hat{\sigma}^3_{L_i}$, as in that case we choose one out of ions i and $i+1$, for example the i^{th}, and induce a Stark-shift of its internal frequency by the application of a laser field Δ-detuned from the carrier transition, $\omega = \omega_0 - \Delta$. The interaction Hamiltonian in the interaction picture and rotating-wave approximation (RWA), with the condition $\omega_0 \gg \nu \gg \Delta$, then reads: $\hat{H}'_{int_i} = \hbar g[\hat{\sigma}^+_i e^{i(\Delta t+\phi)} + H.C.]$, where g is the dipole-laser coupling (Rabi frequency) and ϕ is the laser phase. When the system is in the dispersive regime, $g/|\Delta| \ll 1$, a calculation up to second order in perturbation theory yields for the effective interaction Hamiltonian: $\hat{H}'_{eff_i} = \hbar\frac{g^2}{\Delta}\hat{\sigma}^3_i = B^3_i\hat{\sigma}^3_{L^i}$, with $B^3_i \equiv \hbar g^2/\Delta$ and where we have taken $\delta_i = 1$ in Eq. (1), thus implementing the desired logical operator.

For the implementation of the other Pauli logical operators we consider a laser shining simultaneously on both the i^{th} and the $(i+1)^{th}$ ions, and Δ-detuned from a sideband transition, for example a red-sideband transition (the blue one would work as well), $\omega = \omega_0 - \nu - \Delta$. We assume further that the system is in the Lamb-Dicke regime $\eta \equiv k\sqrt{\hbar/2M\nu} \ll 1$, where η is the Lamb-Dicke parameter, k is the component of the laser field wave vector along the ion motion and M is the total mass of all N ions. The interaction Hamiltonian in the interaction picture and RWA now is: $\hat{H}'_{int_{ii+1}} = \hbar g\eta[\hat{\sigma}^+_i \hat{a} e^{i(\Delta t+\phi+\theta/2)} + \hat{\sigma}^+_{i+1}\hat{a}e^{i(\Delta t+\phi-\theta/2)} + H.C.]$; where $\hat{a}$ is the creation operator of one phonon of the center-of-mass motional mode and $\theta \equiv kz_0$, being z_0 the equilibrium distance between both ions. Once again second order perturbation theory leads to the effective Hamiltonian:

$$\hat{H}'_{eff_{i,i+1}} = \hbar\frac{(g\eta)^2}{\Delta}\{\hat{I} + (\hat{\sigma}^z_i \oplus \hat{\sigma}^z_{i+1})(\hat{a}^\dagger\hat{a} + 1/2) + [\hat{\sigma}^+_i \otimes \hat{\sigma}^-_{i+1}e^{i\theta} + H.C.]\}. \tag{4}$$

The identity operator can be omitted as it only generates an irrelevant global phase factor, and so can the second term, for it is proportional to the z component of the total angular momentum operator, and therefore has no support on $\mathbb{V}_{DFS_2}$. We thus see that the effective interaction Hamiltonian on $\mathbb{V}_{DFS_2}$ is

$$\begin{aligned}\hat{H}'_{eff_{i,i+1}} = \hbar\frac{(g\eta)^2}{\Delta}[\hat{\sigma}^+_i \otimes \hat{\sigma}^-_{i+1}e^{i\theta} + H.C.] \quad &= \quad \hbar\frac{(g\eta)^2}{\Delta}\{\cos(\theta)[\hat{\sigma}^+_i \otimes \hat{\sigma}^-_{i+1} + H.C.] \\ + \sin(\theta)[i\hat{\sigma}^+_i \otimes \hat{\sigma}^-_{i+1} - H.C.]\} \quad &= \quad B^1_i\hat{\sigma}^1_{L_i} + B^2_i\hat{\sigma}^2_{L_i}\,,\end{aligned} \tag{5}$$

with $B^1_i \equiv \hbar\frac{(g\eta)^2}{2\Delta}\cos(\theta)$ and $B^2_i \equiv \hbar\frac{(g\eta)^2}{2\Delta}\sin(\theta)$, and where we have taken $\beta_i = \gamma_i = \delta_i = 1/2$ in Eq. (1). By controlling the value of θ one can produce any desired combination of the logical $\hat{\sigma}^1$ and $\hat{\sigma}^2$ operators, thus completing the implementation of all operators needed to realize Hamiltonian (2). A crucial point in the above derivation was the restriction to the subspace $\mathbb{V}_{DFS_2}$, which led to the vanishing of the z component of the total angular momentum in Eq. (4), allowing us to neglect the terms depending on the vibrational operators. Our derivation can however be extended so as to obtain a two-qubit effective Hamiltonian valid in the full DFS $\mathbb{V}_{DFS_4}$, which is needed, as shown in the next paragraph, to implement two-qubit gates. In order to realize this extension, we notice that if Hamiltonian (4) is added to itself with the replacements $\Delta \leftrightarrow -\Delta$ and $\theta \leftrightarrow \theta + \pi$, one obtains exactly twice Hamiltonian (5), with no dependence on the vibrational mode, even if the z component of the total angular momentum of both qubits is different from zero. Now, the latter is the effective Hamiltonian of any two ions shined on by a bichromatic laser field with one frequency Δ-detuned from a sideband transition and the other $-\Delta$-detuned from the same transition and with both wave vectors, k_1 and k_2, chosen so that $\theta_1 = k_1 z_0 = \theta$ and $\theta_2 = k_2 z_0 = \theta + \pi$. One eliminates in this way the motional degree of freedom for the whole DFS.

We choose now four out of the N ions, the i^{th}, the $(i+1)^{th}$, the j^{th} and the $(j+1)^{th}$ ones, to compose the i^{th} and j^{th} logical qubits. A $\hat{\sigma}^3_i \otimes \hat{\sigma}^3_j$ interaction between any ion from the i^{th} pair and any one from the j^{th} pair is required to realize the interaction term of Hamiltonian (3). Nevertheless, such an effective interaction is not achievable with laser-fields manipulating the internal states of ions. It is impossible to realize a non-local gate between two logical qubits using only two-body interactions, under the requirement that the states involved in the operation stay in the encoded subspace $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$.

This is however too strong a requirement. In fact, in order to implement a fault-tolerant gate it is sufficient that the states remain in the DFS supported by all four ions $\mathbb{V}_{DFS_4}$. This is accomplished by using the encoding recoupling scheme originally developed in [7] for NMR systems. This scheme is based on the following identity: $e^{-i[\hat{\sigma}^+_i \otimes \hat{\sigma}^-_j + H.C.]\pi/4} e^{-i[\hat{\sigma}^+_i \otimes \hat{\sigma}^-_{i+1} + H.C.]\pi/2} [\hat{\sigma}^+_{i+1} \otimes \hat{\sigma}^-_j +$

$H.C.\big] e^{i[\hat{\sigma}_i^+ \otimes \hat{\sigma}_{i+1}^- + H.C.]\pi/2} e^{i[\hat{\sigma}_i^+ \otimes \hat{\sigma}_j^- + H.C.]\pi/4} \equiv \frac{1}{2}\hat{\sigma}_{i+1}^3 \otimes \left(\hat{\sigma}_j^3 - \hat{\sigma}_i^3\right)$. Since $\hat{\sigma}_{i+1}^3 \otimes \hat{\sigma}_i^3$ on the right-hand side of the last equality is proportional to the identity operator in the encoded space $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$, it can be ignored, thus leaving us with the Hamiltonian $\frac{1}{2}\hat{\sigma}_{i+1}^3 \otimes \hat{\sigma}_j^3 \equiv -\frac{1}{2}\hat{\sigma}_{L_i}^3 \otimes \hat{\sigma}_{L_j}^3$, which is related to Hamiltonian (3) by local logical transformations. As our method involves (slow) Raman transitions and is used several times in the encoding recoupling scheme, the duration of the procedure must be compared to realistic entanglement lifetimes. For the experimental values at the Innsbruck experiment: $g = 2\pi \times 120KHz$, $\eta = 0.02$ and $\Delta = 2\pi \times 120KHz$, the time required to realize, for instance, the operation $e^{-i[\hat{\sigma}_n^+ \otimes \hat{\sigma}_m^- + H.C.]\pi/2}$ is $\tau \equiv \frac{\pi\Delta}{4(g\eta)^2} = 2.6$ ms, which is four orders of magnitude smaller than the 20 seconds robust entanglement reported in [6].

Summary–We have proposed a procedure to demonstrate, with already existing technology, fault-tolerant universal quantum gates on DFS's with ions trapped in a linear Paul trap, using interactions between only two ions at a time. Individual laser-addressing to each ion is required, but the ions do not need to be in their motional ground state provided that they always remain in the Lamb-Dicke regime. The proposal applies the encoding recoupling scheme originally developed in [7] for NMR systems, in which both logical qubits momentarily "loose their encoded identity" by exiting $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$ but not the total DFS supported by all four qubits, $\mathbb{V}_{DFS_4}$. A necessary condition for the latter to be a protected subspace is that all four qubits experience the same fluctuations, which is a realistic assumption as long as they are not far apart, as compared to the typical coherence length of the fluctuations. For systems in which the latter assumption cannot be taken [8], preserving the state of the system within $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$ throughout any computation becomes crucial. Therefore, and after presenting the general two-physical-qubit logical Pauli operators, we also derived the most general two-physical-qubit interaction Hamiltonian that supports fault-tolerant universal quantum computation on any DFS, without ever exiting $\mathbb{V}_{DFS_2} \otimes \mathbb{V}_{DFS_2}$.

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