Quantum information exchange between three coupled modes

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Abstract

The problem of quantum state exchange between coupled modes of the electromagnetic field modelled by quantum oscillators are considered. Analyzing the structure of propagators of the Schrödinger equation with the most general weak bilinear resonance coupling, we found some conditions of state exchange for arbitrary initial states for two and a particular case of three coupled oscillators. At this moment we are giving attention to the case of three coupled oscillators in a linear chain of oscillators by considering the most general time *independent* bilinear coupling between them.

The problem of so-called "quantum teleportation", or a transfer of the state of some quantum system to another quantum system has attracted attention of many authors. There exist various schemes [1–5], some of which have been realized already [6,7]. The aim of our study is to consider the possibilities of the *quantum state exchange* between three or more coupled modes of the electromagnetic field modelled by coupled quantum oscillators as an extension for the case of two coupled modes considered in [8]. It is well known from classical mechanics that two weakly coupled identical oscillators periodically exchange their energies. It is true that the oscillators exchange not only energies, but also their *quantum states* and such an exchange can be interpreted also as an ideal *information transfer* [9]. Thus a simple question arise in how we can control this information exchange between more than two modes. In some special cases (e.g., for particular initial states, such as squeezed states or coherent states and their "cat" superpositions, or for some specific couplings between the modes) this problem was studied recently in [9–12] but only for two modes - bipartite systems. Our goal is to consider the more general cases of multipartite system starting our study with the more simple case of three coupled modes as our "multipartite" system. The more general case of three coupled modes we can study is described by the total hamiltonian

$$H = \frac{\hbar\omega_1}{2}(p_1^2 + x_1^2) + \frac{\hbar\omega_2}{2}(p_2^2 + x_2^2) + \frac{\hbar\omega_3}{2}(p_3^2 + x_3^2) + \epsilon\hbar\varpi_1(\gamma_1 p_1 p_2 + \gamma_2 p_1 x_2 + \gamma_3 p_1 x_2 + \gamma_4 x_1 x_2)$$
$$+\epsilon\hbar\varpi_2(\gamma_5 p_1 p_3 + \gamma_6 p_1 x_3 + \gamma_7 x_1 p_3 + \gamma_8 x_1 x_3) + \epsilon\hbar\varpi_3(\gamma_9 p_2 p_3 + \gamma_{10} p_2 x_3 + \gamma_{11} x_2 p_3 + \gamma_{12} x_2 x_3) \quad (1)$$

where p_i and x_i are dimensionless quadrature components variables, whose operators obey the commutation relations $[\hat{x}_j, \hat{p}_k] = i\delta_{jk}$, $\varpi_1 \equiv \sqrt{\omega_1\omega_2}$, $\varpi_2 \equiv \sqrt{\omega_1\omega_3}$ and $\varpi_3 \equiv \sqrt{\omega_2\omega_3}$. The dimensionless coupling constants $\epsilon\gamma$ are time independent. In the case of material oscillators with the masses m_j, the dimension variables P_i and X_i are related to their dimensionless partners as follows: $p_i = P_i/\sqrt{m_i\omega_i\hbar}$, $x_i = X_i\sqrt{m_i\omega_i/\hbar}$. The Hamiltonian (1) is a special case of generic quadratic Hamiltonians, which can be written as (for the sake of simplicity, we confine ourselves to the homogeneous Hamiltonian with$m_i = 1$, $\hbar = 1$)

$$H = \frac{1}{2}\mathbf{q}\mathcal{B}\mathbf{q}, \quad \mathbf{q} \equiv (\mathbf{p}, \mathbf{x}), \quad (2)$$

where $\mathbf{x}$ is N-dimensional "coordinate" vector and $\mathbf{p}$ is the canonically conjugated "momentum" vector. $\mathcal{B}(t)$ is a *symmetrical* $2N \times 2N$ matrix

$$\mathcal{B} = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}, \quad (3)$$

consisting of four $N \times N$ blocks satisfying the conditions $b_1 = \tilde{b}_1$, $b_4 = \tilde{b}_4$, $b_2 = \tilde{b}_3$ (tilde means matrix transposition). The blocks of Λ-matrix

$$\Lambda(t) = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}, \quad (4)$$

determine the propagator of the Schrödinger equation with the Hamiltonian (2) in the coordinate representation [14]:

$$G\left(\mathbf{x}_2;\mathbf{x}_1;t\right)=\left[\det\left(-2\pi i\lambda_3\right)\right]^{-1/2}\exp\left\{-\frac{i}{2}\left[\mathbf{x}_2\lambda_3^{-1}\lambda_4\mathbf{x}_2-2\mathbf{x}_2\lambda_3^{-1}\mathbf{x}_1+\mathbf{x}_1\lambda_1\lambda_3^{-1}\mathbf{x}_1\right]\right\}. \tag{5}$$

To find the explicit time dependence, one has to solve the matrix equation

$$\dot{\Lambda}=\Lambda\Sigma\mathcal{B},\quad \Sigma=\left[\begin{array}{cc}0 & I_N\\ -I_N & 0\end{array}\right],\quad \Lambda(0)=I_{2N}, \tag{6}$$

where I_N means the $N\times N$ unity matrix. To describe the dynamics of the three coupled oscillators, we have to solve Eq. (6), which is equivalent to the following system of linear differential equations and initial conditions for the matrices $\lambda_j(t)$:

$$\dot{\lambda}_1=\lambda_1\, b_3-\lambda_2\, b_1,\qquad \lambda_1(0)=I_N, \tag{7}$$
$$\dot{\lambda}_2=\lambda_1\, b_4-\lambda_2\, b_2,\qquad \lambda_2(0)=\ 0, \tag{8}$$
$$\dot{\lambda}_3=\lambda_3\, b_3-\lambda_4\, b_1,\qquad \lambda_3(0)=\ 0, \tag{9}$$
$$\dot{\lambda}_4=\lambda_3\, b_4-\lambda_4\, b_2,\qquad \lambda_4(0)=I_N. \tag{10}$$

For nonsingular matrix b_1, one can exclude matrices λ_2 and λ_4, arriving at the identical second-order equations for the matrices λ_3 and λ_1:

$$\frac{d^2\lambda_3}{dt^2}-\frac{d\lambda_3}{dt}G_3+\lambda_3G_4=0,\quad G_3=b_3-b_1^{-1}b_2b_1,\quad G_4=b_4b_1-b_3b_1^{-1}b_2b_1, \tag{11}$$
$$\lambda_1(0)=I_N,\quad \dot{\lambda}_1(0)=b_3,\qquad \lambda_3(0)=0,\quad \dot{\lambda}_3(0)=-b_1. \tag{12}$$

In solving the dynamics equations in the weak coupling limit ($\gamma_i\gamma_j\ll 1$), we apply the multiple scale method [15] to the system of differential equation for the case of three *resonant* ($\omega_1=\omega_2=\omega_3=1$) quantum harmonic oscillators (condition for information exchange). We start by introducing new independent variables according to $T_n=\epsilon^n t,\ n=0,1,2\cdots$. It follows that the derivatives with respect to t become expansion in terms of the partial derivatives with respect to the T_n according to

$$\frac{d}{dt}=\mathcal{D}_0+\epsilon\mathcal{D}_1+\epsilon^2\mathcal{D}_2+\mathcal{O}(\epsilon^3),\quad \frac{d^2}{dt^2}=\mathcal{D}_0^2+2\epsilon\mathcal{D}_1\mathcal{D}_0+\epsilon^2(\mathcal{D}_1^2+2\mathcal{D}_0\mathcal{D}_2)+\mathcal{O}(\varepsilon^3). \tag{13}$$

One assumes that the solution of the system of second order differential equations (11) write in the form

$$\ddot{\lambda}+\lambda\mathcal{M}_0=\epsilon\left[\dot{\lambda}\mathcal{F}+\lambda\mathcal{K}\right],\qquad \mathcal{M}_0=\mathrm{diag}\left(1,1,1\right), \tag{14}$$

with the anti-diagonal matrices in the right-hand side in the form

$$\mathcal{F}=-\left[\begin{array}{ccc}0 & \nu_1 & \nu_2\\ -\nu_1 & 0 & \nu_3\\ -\nu_2 & -\nu_3 & 0\end{array}\right],\quad \mathcal{K}=-\left[\begin{array}{ccc}0 & \mu_1 & \mu_2\\ \mu_1 & 0 & \mu_3\\ \mu_2 & \mu_3 & 0\end{array}\right], \tag{15}$$

with μ_i and ν_i defined in terms of coupling constants γ_k with the help of (11). The λ_k matrices in (4) has the form

$$\lambda_k=\lambda_{k,0}(T_0,T_1,T_2,...)+\epsilon\lambda_{k,1}(T_0,T_1,T_2,...)+\epsilon^2\lambda_{k,2}(T_0,T_1,T_2,...)\cdots \tag{16}$$

We note that the number of independent time scale needed depends on the order to which the expansion is carried out. In scaled variable the system to be solved assume the form

$$D_0^2\lambda_{k,0}+\lambda_{k,0}\mathcal{M}=0, \tag{17}$$

$$\mathcal{D}_0^2\lambda_{k,1}+\lambda_{k,1}\mathcal{M}=\mathcal{D}_0\lambda_{0,k}\mathcal{F}-2\mathcal{D}_1\mathcal{D}_0\lambda_{k,0}+\lambda_{k,0}\mathcal{K}, \tag{18}$$

$$\mathcal{D}_0^2\lambda_{k,2}+\lambda_{k,2}\mathcal{M}=-\mathcal{D}_1^2\lambda_{k,0}-2\mathcal{D}_1\mathcal{D}_0\left(\lambda_{k,0}+\lambda_{k,1}\right)+\left(\mathcal{D}_0\lambda_{0,k}+\mathcal{D}_1\lambda_{1,k}\right)\mathcal{F}+\lambda_{k,1}\mathcal{K}, \tag{19}$$

with the initial conditions determined from the equations

$$\lambda_k(0) = \lambda_{k,0}(0,0,0) + \epsilon\lambda_{k,1}(0,0,0) + \epsilon^2\lambda_{k,2}(0,0,0) + \cdots \tag{20}$$

$$\dot{\lambda}_k(0) = \mathcal{D}_0\lambda_{k,0}|_{(0,0,0)} + \epsilon\left(D_0\lambda_{k,1} + D_1\lambda_{k,0}\right)|_{(0,0,0)} + \epsilon^2\left(D_0\lambda_{k,2} + D_1\lambda_{k,1} + D_2\lambda_{k,0}\right)|_{(0,0,0)} + \cdots \tag{21}$$

This is the system we use to find the Λ matrix solution in order to describe the approximated behavior of the system, a well defined solution valid in the weak coupling limit, $\gamma_k\gamma_j \ll 1$. From the solution of λ matrix we find the explicit form of the propagator (5). We assume that the initial wave function of the total system is factorized,

$$\psi\left(x_1, x_2, x_3; 0\right) = \psi_1\left(x_1; 0\right)\psi_2\left(x_2; 0\right)\psi_3\left(x_3; 0\right). \tag{22}$$

Then the wave-function at any moment of time $t > 0$ can be written as

$$\psi\left(x_1, x_2, x_3; t\right) = \int G\left(x_1, x_2, x_3; x_1', x_2', x_2'; t\right)\psi\left(x_1', x_2', x_2'; 0\right) dx_1' dx_2' dx_3'. \tag{23}$$

At the initial moment $t = 0$ one has

$$G\left(x_1, x_2, x_3; x_1', x_2', x_3'; 0\right) = \delta\left(x_1 - x_1'\right)\delta\left(x_2 - x_2'\right)\delta\left(x_3 - x_3'\right).$$

In [8] two kind of state exchange for a bipartite system was defined: a narrow sense, the state exchange means that at some instant of time t_* the total wave function is factorized again into a product of the initial terms, but with exchanged arguments:

$$\psi\left(x_1, x_2; t_*\right) = \mathcal{N}\psi_1\left(s_1 x_2; 0\right)\psi_2(s_2 x_1; 0), \tag{24}$$

with some scaled by factors $s_{2,1}$and some phase factor $\mathcal{N}$. To ensure (24) the propagator at the instant t_* must turn into a product of two delta-functions as

$$G\left(x_1, x_2; x_1', x_2'; t_*\right) = \mathcal{N}\delta\left(s_1 x_2 - x_1'\right)\delta\left(s_2 x_1 - x_2'\right). \tag{25}$$

The other kind, the state exchange in a wide sense" or "state exchange of the second kind" arises when

$$\psi\left(x_1, x_2; t_*\right) = \mathcal{N}\psi_1\left(s_1 x_2; t_1\right)\psi_2\left(s_2 x_1; t_2\right), \tag{26}$$

which in the propagator language condition (26) reads as

$$G\left(x_1, x_2; x_1', x_2'; t_*\right) = \mathcal{N}G_1^{(f)}\left(s_1 x_2; x_1'; t_1\right)G_2^{(f)}\left(s_2 x_1; x_2'; t_2\right) \tag{27}$$

where $G_k^{(f)}\left(x; x'; t\right)$ is the free propagator of the isolated kth mode.

In the case of three weak coupling resonant oscillators similar results were found. Initially we considered a special condition: $\mu_2 = \nu_2 = 0$, which means the mode 1 and 3 being uncoupled or in a situation of wave rotating coupling. In order to determine when some state exchange occurs we define the coupling intensity ρ_k and phases ϕ_k by the following equations

$$\mu_1 + i\nu_1 = \rho_1\exp(i\phi_1), \quad \mu_3 + i\nu_3 = \rho_3\exp(i\phi_3) \tag{28}$$

We also have now the scaled slow time τ, given by $\tau = \rho t/2 = \sqrt{(\mu_1^2 + \mu_2^2 + \nu_1^2 + \nu_2^2)}t/2$ and the natural time t treated as independent variable, as prescribed by the multiple scale method. By following the same procedure as done in [8] we write explicitly the λ_3 determinant and find the conditions under which we have $\parallel \lambda_3 \parallel = 0$. This first step give a pre-factored form of the propagator indicating the exact time when the factorization of the form (25) or (27) occurs. For example, in the case of

$$\rho_1^2\cos^2\tau + \rho_3^2 = 0 \tag{29}$$

with $\rho_1 = \rho_3$ we find a state exchange of the first occurring at the instant of $t_m = m\pi$, with the phase $\phi_1 + \phi_3 = 0$ and $\theta_m = (-1)^m$:

$$\psi(x_1, x_2, x_3, t_{2m+1}) = \psi_1(-\theta_m x_3)\psi_2(-\theta_m x_2)\psi_3(-\theta_m x_1) \tag{30}$$

In a similar way for $\phi_1 + \phi_3 = \pi/2$ we found

$$\psi(x_1, x_2, x_3, t_m) = \psi_1(\theta_m x_3)\psi_2(\theta_m x_2)\psi_3(-\theta_m x_1) \tag{31}$$

All other possibilities were considered. Then in a chain of three coupling quantum oscillator (in a line) this results shows that we can exchange quantum information (in a narrow sense) between the mode 1 and 3 (the extremes mode) without *no quantum exchange* in the mode 2 (intermediary mode). This was done for arbitrary initial states for each modes - no consideration about the nature of the quantum states was initially considered. As a starting point for our generalization to N coupled mode this allows to the question: How can we control the coupling parameters in quantum chain of quantum oscillator in order to transmit a general quantum information between the first and the "last" mode? This we hope to answer.

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