Generating Superpositions of Coherent States in the Detuned *Jaynes-Cummings* Model with a *Kerr*-like Medium

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Abstract

We investigate the dynamics of the *Jaynes-Cummings* model with a *Kerr*-like medium seeking the generation and retrieval of superpositions of coherent states. The method depends on the choice of suitable combinations of the atom-field detuning δ and nonlinear susceptibility $\chi^{(3)}$ that yield a linear *Rabi* frequency and a periodic dynamics. For the field initially in a coherent state and the atom excited we have found a way to control the atom-field disentanglement at each revival time. Moreover, quantum superpositions of coherent states or *Schrödinger* cats may be generated at each collapse time for all suitable combinations, each one with a specific relative phase depending on the values of δ and $\chi^{(3)}$. For the field initially in a statistical mixture of two coherent states, we show that two of such combinations yield pure states (*Schrödinger* cats) at each collapse time. We present these results analytically and numerically considering the field linear entropy, the *Wigner* function and the field fidelity.

1 Introduction

The *Jaynes-Cummings* model (**JCM**) describes the interaction between a two-level atom and a single quantized mode of the electromagnetic field in a lossless cavity within the rotating wave approximation (**RWA**) being the simplest model of the radiation-matter interaction and having exactly integrable solutions [1, 2]. During almost forty years, this model have been studied from various interesting aspects, being a basis for more complicated and generalized models [3].

With the recent refinement of experimental techniques in quantum optics, the generation of macroscopic quantum superpositions (*Schödinger* cats) have gained considerable interest. States of this sort have been generated in different contexts [4]. However, optical *Schrödinger* cats have not yet been realized. A great variety of methods have been proposed for generate such states, for example, a recent proposal in a *Mach-Zehnder* interferometer [5] and one of the first schemes due to *Yurke* and *Stoler* [6] who showed that a coherent state (**CS**) propagating in a *Kerr* medium could lead to a superposition of **CS**s of the form $|\alpha\rangle_{YS} = \frac{1}{\sqrt{2}}(|\alpha\rangle + \mathrm{e}^{i\vartheta}|-\alpha\rangle)$, where $\vartheta = \pi/2$ (the so-called *Yurke-Stoler* **CS**). In this work we present the **JCM** with a *Kerr* medium and show that is possible generate superpositions of **CS**s. Moreover, the initial system state can be retrieved. As far as we are aware, the method has not previously been sufficiently explored in the literature.

2 The Model

We consider a two-level atom interacting with a high-Q single-mode cavity filled with a nonlinear *Kerr* medium (modelled here as an anharmonic oscillator). Supposing the adiabatic limit for the *Kerr* medium [7] and the dipole and **RWA** approximations, the total Hamiltonian of this system can be written as [8]

$$\boldsymbol{H} = \hbar\omega_0 \boldsymbol{a}^\dagger \boldsymbol{a} + \tfrac{1}{2}\hbar\omega_{eg}\boldsymbol{\sigma}_z + \hbar\chi^{(3)}\boldsymbol{a}^{\dagger 2}\boldsymbol{a}^2 + \hbar\Omega\big(\boldsymbol{a}^\dagger\boldsymbol{\sigma}_- + \boldsymbol{a}\boldsymbol{\sigma}_+\big), \tag{1}$$

where ω_0 and ω_{eg} are, respectively, the cavity field and atomic transition frequencies, $\boldsymbol{a}^\dagger$ ($\boldsymbol{a}$) is the creation (annihilation) operator of the cavity mode, $\boldsymbol{\sigma}_z$, $\boldsymbol{\sigma}_+$ and $\boldsymbol{\sigma}_-$ are the *Pauli* matrices operators, with $|e\rangle$ ($|g\rangle$) being the excited (ground) state for the atom, Ω is the atom-field coupling constant and $\chi^{(3)}$ is the anharmonicity parameter, proportional to the third-order susceptibility.

Considering the *Stenholm* approach [9] we obtain, after some algebra, the exact (**RWA**) solution of the time evolution operator for this model

$$\boldsymbol{U}_I(t) = \mathrm{e}^{-\frac{i}{2}\chi^{(3)}t}\mathrm{e}^{-i\chi^{(3)}(\boldsymbol{n}^2-\boldsymbol{n})t}\begin{pmatrix}\mathrm{e}^{-i\chi^{(3)}(\boldsymbol{n}-\frac{1}{2})t} & 0\\ 0 & \mathrm{e}^{i\chi^{(3)}(\boldsymbol{n}-\frac{1}{2})t}\end{pmatrix}$$

$$\times\begin{pmatrix}\cos\left(\frac{1}{2}\boldsymbol{\Omega}_{n+1}t\right)-i\boldsymbol{\gamma}_{n+1}\dfrac{\sin\left(\frac{1}{2}\boldsymbol{\Omega}_{n+1}t\right)}{\boldsymbol{\Omega}_{n+1}} & -2i\Omega\boldsymbol{a}\dfrac{\sin\left(\frac{1}{2}\boldsymbol{\Omega}_{n}t\right)}{\boldsymbol{\Omega}_{n}}\\ -2i\Omega\boldsymbol{a}^\dagger\dfrac{\sin\left(\frac{1}{2}\boldsymbol{\Omega}_{n+1}t\right)}{\boldsymbol{\Omega}_{n+1}} & \cos\left(\frac{1}{2}\boldsymbol{\Omega}_{n}t\right)+i\boldsymbol{\gamma}_{n}\dfrac{\sin\left(\frac{1}{2}\boldsymbol{\Omega}_{n}t\right)}{\boldsymbol{\Omega}_{n}}\end{pmatrix}, \tag{2}$$

where

$$\boldsymbol{\Omega}_{n+1} = \sqrt{\boldsymbol{\gamma}_{n+1}^2 + 4\Omega^2(\boldsymbol{n}+1)}, \tag{3}$$

with $\boldsymbol{\gamma}_{n+1} = \delta - 2\chi^{(3)}\boldsymbol{n}$.

2.1 Linear Rabi Frequency

To examine analytically the effects of δ and $\chi^{(3)}$ on the *Rabi* frequency, we expand the eigenvalues of Eq. (3) in $\bar{n}$ and derive the first two terms to obtain the revival and super-revival times

$$t_r = \frac{2\pi}{\left|(\dot{\Omega}_{n+1})_{n=\bar{n}}\right|} = \pi\left|\frac{\Omega_{\bar{n}+1}}{\Delta_{\bar{n}+1}}\right| \quad\text{and}\quad t_s = \frac{2\pi}{\left|(\frac{1}{2}\ddot{\Omega}_{n+1})_{n=\bar{n}}\right|} = \pi\left|\frac{\Omega^3_{\bar{n}+1}}{\chi^{(3)2}\Omega^2_{\bar{n}+1}-\Delta^2_{\bar{n}+1}}\right|, \tag{4}$$

where $\Delta_{\bar{n}+1} = \Omega^2 - \chi^{(3)}\gamma_{\bar{n}+1}$. When $\Delta_{\bar{n}+1} = \chi^{(3)}\Omega_{\bar{n}+1}$, we have $\ddot{\Omega}_{\bar{n}+1} = 0$ and, hence, $t_s \to \infty$. This condition is given by $\delta_c = \frac{\Omega^2}{2\chi^{(3)}} - 2\chi^{(3)}$ and holds for every $\bar{n}$, yielding a linear *Rabi* frequency $\Omega_{n+1} = \delta_c + 2\chi^{(3)}(n+2)$ and a periodic dynamics [10]. We comment now the **RWA** validity: from experimental parameters [11], $\Omega \sim 10^4\,\mathrm{Hz}$, $\omega_0 \sim 10^{10}\,\mathrm{Hz}$ (microwave frequency) and then $\delta \sim 10^2\Omega \sim 10^{-4}\omega_0 \ll \omega_0$; hence, the **RWA** holds.

3 Field Dynamics

Considering the atom-field state initially uncorrelated described by the product state $\boldsymbol{\rho} = \boldsymbol{\rho}_a \otimes \boldsymbol{\rho}_f$ we obtain its evolution from $\boldsymbol{\rho}(t) = \boldsymbol{U}_I(t)\boldsymbol{\rho}\boldsymbol{U}_I^\dagger(t)$. Choosing the atom initially excited or $\boldsymbol{\rho}_a = |e\rangle\langle e|$ and two different initial field states: a **CS** or $\boldsymbol{\rho}_C = |\alpha\rangle\langle\alpha|$ and a statistical mixture of two **CS**s or $\boldsymbol{\rho}_{mix} = \frac{1}{2}\big(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|\big)$, we can write the evolution of the reduced field density operator as

$$\boldsymbol{\rho}_f(t) = \mathrm{Tr}_a\big[\boldsymbol{\rho}(t)\big] = \sum_{n,m}\rho_{n,m}\mathrm{e}^{-i\chi^{(3)}(n^2-m^2)t}\big(A_{n+1}A^*_{m+1}|n\rangle\langle m| + B_{n+1}B^*_{m+1}|n+1\rangle\langle m+1|\big), \tag{5}$$

where $A_{n+1} = C_{n+1} - i\gamma_{n+1}S_{n+1}$ and $B_{n+1} = 2i\Omega\sqrt{n+1}S_{n+1}$ with $C_{n+1} = \cos\left(\frac{1}{2}\Omega_{n+1}t\right)$ and $S_{n+1} = \sin\left(\frac{1}{2}\Omega_{n+1}t\right)/\Omega_{n+1}$ and $\rho_{n,m} = \langle n|\boldsymbol{\rho}_f|m\rangle$ are the initial field matrix elements.

A very useful operational measure of the field state purity is given by the field linear entropy

$$\zeta_f(t) = 1 - \mathrm{Tr}_f\big[\boldsymbol{\rho}_f^2(t)\big] = 1 - \sum_n\langle n|\boldsymbol{\rho}_f^2(t)|n\rangle = 1 - \sum_{n,m}\left|\rho_{n,m}(t)\right|^2, \tag{6}$$

where $\rho_{n,m}(t) = \langle n|\boldsymbol{\rho}_f(t)|m\rangle$ are the evolved field matrix elements.

When $\delta = \delta_c$, we observe that both initial states of the field yield a *Schrödinger* cat at each collapse time and are retrieved at each revival time (center and right plots in Fig. 1). These states may be "visualized" considering the *Wigner* function [12]

$$W(\beta,t) = \frac{2}{\pi}\sum_n(-1)^n\langle n|\boldsymbol{D}^\dagger(\beta)\boldsymbol{\rho}(t)\boldsymbol{D}(\beta)|n\rangle = \frac{2}{\pi}\sum_{n,m}(-1)^n\rho_{n,m}(t)\langle m|\boldsymbol{D}(2\beta)|n\rangle, \tag{7}$$

and might be probed using the field fidelity, defined as

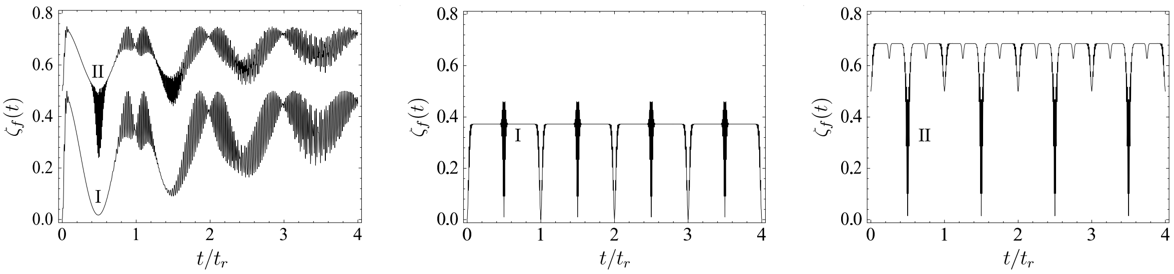

Figure 1: Field linear entropy as a function of t/t_r for the field initially in (I) $\boldsymbol{\rho}_C = |\alpha\rangle\langle\alpha|$ or (II) $\boldsymbol{\rho}_{mix} = \frac{1}{2}(|\alpha\rangle\langle\alpha| + |-\alpha\rangle\langle-\alpha|)$ ($\alpha = 5$, real) and for the atom initially in $\boldsymbol{\rho}_a = |e\rangle\langle e|$ with $\delta = 0$ and $\chi^{(3)} = 0$ (left plot) and with $\delta = 4.8\Omega$ and $\chi^{(3)} = 0.1\Omega$ (center and right plots).

$$\mathcal{F}_f(t) = \mathrm{Tr}_f\left[\boldsymbol{\rho}_f(t)\boldsymbol{\rho}_f\right] = \sum_{n,m} \rho_{n,m}(t)\rho_{m,n}. \tag{8}$$

When $\delta = \delta_c$ the collapse time will be given by $\frac{1}{2}t_r = \pi/2\chi^{(3)}$. Using Eq. (8), we compare the evolutions that could lead to the pure state

$$|\beta;\vartheta\rangle = \mathcal{C}^{\frac{1}{2}}\left(|\beta\rangle + \mathrm{e}^{i\vartheta}|-\beta\rangle\right), \tag{9}$$

where $\mathcal{C} = \frac{1}{2}\left(1 + \mathrm{e}^{-2|\alpha|^2}\cos\vartheta\right)^{-1}$, and $\beta = |\beta|\mathrm{e}^{i\phi}$. From the results addressed in Ref. [13] we see that a **CS** with an initial phase $\phi = 0$ evolves to a superposition like in Eq. (9) having $\phi = \pi/2$ for both $\boldsymbol{\rho}_C$ and $\boldsymbol{\rho}_{mix}$ initial field states. We then scan the phase ϑ at the collapse time until obtain $\mathcal{F}_f(\frac{1}{2}t_r) \approx 1$. The results for $\boldsymbol{\rho}_C$ with different combinations of δ and $\chi^{(3)}$ are displayed in Tab. 1. We observe that each one of the combinations with $\delta = \delta_c$ yields a *Schrödinger* cat at the collapse time with a specific relative phase ϑ and that the initial state $\boldsymbol{\rho}_C = |-\alpha\rangle\langle-\alpha|$ yields the same pure state at the collapse time but with phase $-\vartheta$. We pay special attention to two of these combinations: ($\delta \approx 4.474\Omega$, $\chi^{(3)} = \frac{8}{75}\Omega$) which generates an odd **CS** ($\vartheta = \pi$) and ($\delta = 4.8\Omega$, $\chi^{(3)} = 0.1\Omega$) which generates an even **CS** ($\vartheta = 0$). When the field is initially in $\boldsymbol{\rho}_{mix}$ only for these special combinations we have the generation of *Schrödinger* cats and observe that they are the same as obtained for $\boldsymbol{\rho}_C$ (center and right plots in Fig. 1). To understand these results we see that the density operators

$$\begin{aligned}\boldsymbol{\rho}_{CS}^{pure} &= |\beta;\vartheta\rangle\langle\beta;\vartheta| = \mathcal{C}\left[|\beta\rangle\langle\beta| + |-\beta\rangle\langle-\beta| + \cos\vartheta(|\beta\rangle\langle-\beta| + |-\beta\rangle\langle\beta|) - i\sin\vartheta(|\beta\rangle\langle-\beta| - |-\beta\rangle\langle\beta|)\right],\\ \boldsymbol{\rho}_{CS}^{mix} &= \tfrac{1}{2}\left(|\beta;\vartheta\rangle\langle\beta;\vartheta| + |\beta;-\vartheta\rangle\langle\beta;-\vartheta|\right) = \mathcal{C}\left[|\beta\rangle\langle\beta| + |-\beta\rangle\langle-\beta| + \cos\vartheta(|\beta\rangle\langle-\beta| + |-\beta\rangle\langle\beta|)\right],\end{aligned} \tag{10}$$

are equals only if $\sin\vartheta = k\pi$ (k integer), i.e., only for the even and odd **CS**s.

Table 1: Field fidelity at time $\frac{1}{2}t_r$ for different combinations of δ and $\chi^{(3)}$ with the respective ϑ.

δ	0	0	0.45Ω	$\frac{16}{15}\Omega$	2.1Ω	4.474Ω	4.8Ω	9.9Ω	49.98Ω	99.99Ω
$\chi^{(3)}$	0	0.5Ω	0.4Ω	0.3Ω	0.2Ω	$\frac{8}{75}\Omega$	0.1Ω	0.05Ω	0.01Ω	0.005Ω
ϑ	1.21π	0.45π	0.52π	0.39π	0.70π	π	0	1.24π	1.49π	1.53π
$\mathcal{F}_f(\frac{1}{2}t_r)$	0.7829	0.9674	0.9416	0.9228	0.8742	0.9861	0.9899	0.9307	0.9897	0.9946

Now we present the analytical results in the dispersive limit, i.e., at high detuning interaction. Following the usual procedure to obtain the dispersive approximation of the **JCM** [14] and analogously to the procedure described above we can write the evolved field state as

$$|\psi_f(t)\rangle = \sum_n c_n \mathrm{e}^{-i\chi^{(3)}n^2 t}\mathrm{e}^{i\chi^{(3)}nt}\mathrm{e}^{-i\frac{\Omega^2}{\delta}nt}|n\rangle, \tag{11}$$

where $c_n = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\frac{\alpha^n}{\sqrt{n!}}$ gives the **CS** statistics. When $\delta = \delta_c$, at the revival time $t_r = \pi/\chi^{(3)}$ the initial state will be retrieved

$$|\psi_f(t_r)\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{1}{\sqrt{n!}}\left(\mathrm{e}^{i\pi}\mathrm{e}^{-i\pi\frac{\Omega^2}{\delta\chi^{(3)}}}\alpha\right)^n \mathrm{e}^{-i\pi n^2}|n\rangle = |\tilde{\alpha}\rangle \approx |\alpha\rangle, \tag{12}$$

where we have used $\delta = 49.98\Omega$ and $\chi^{(3)} = 0.01\Omega$. We remark that this result can by easily generalized for all initial field states. Analogously, at the collapse time $\frac{1}{2}t_r = \pi/2\chi^{(3)}$ we have

$$|\psi_f(\tfrac{1}{2}t_r)\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2} \sum_n \frac{1}{\sqrt{n!}} \big(\mathrm{e}^{i\frac{\pi}{2}} \mathrm{e}^{-i\frac{\pi}{2}\frac{\Omega^2}{\delta\chi^{(3)}}} \alpha\big)^n \mathrm{e}^{-i\frac{\pi}{2}n^2} |n\rangle. \quad (13)$$

Using the identity $\mathrm{e}^{-i\frac{\pi}{2}n^2} = \frac{1}{2}(1+i)(\mathrm{e}^{-i\pi n} - i)$, after multiplying by the global phase $\frac{1}{\sqrt{2}}(1-i)$, we obtain, for $\delta = 49.98\Omega$ and $\chi^{(3)} = 0.01\Omega$

$$|\psi_f(\tfrac{1}{2}t_r)\rangle = \tfrac{1}{\sqrt{2}}\big(|-i\mathrm{e}^{-i\frac{\pi}{2}\frac{\Omega^2}{\delta\chi^{(3)}}}\alpha\rangle - i|i\mathrm{e}^{-i\frac{\pi}{2}\frac{\Omega^2}{\delta\chi^{(3)}}}\alpha\rangle\big) = \tfrac{1}{\sqrt{2}}\big(|i\alpha\rangle - i|-i\alpha\rangle\big), \quad (14)$$

i.e., a superposition of **CS**s in the imaginary axis of the phase space, with a relative phase $\vartheta = -\pi/2$ in agreement to that obtained numerically (Tab. 1).

4 Conclusions

We have observed that the dynamics of the **JCM** with a *Kerr* medium is considerably richer than it is shown in the literature. The parameters δ and $\chi^{(3)}$ may combine in a way that new and interesting features are revealed. In particular, we have a periodic dynamics dictated by δ_c, which allows us to retrieve the initial state at each revival time. We have also found that an initially coherent field generates a superposition of **CS**s at each collapse time. Moreover, we have found the condition for which the field initially in a statistical mixture of two **CS**s evolves towards a pure state, e.g., to an even **CS**.

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