Hamiltonian formulation of nonAbelian noncommutative gauge theories

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Abstract

We implement the Hamiltonian treatment of a nonAbelian noncommutative gauge theory, considering with some detail the algebraic structure of the noncommutative symmetry group. The first class constraints and Hamiltonian are obtained and their algebra derived, as well as the form of the gauge invariance for the first order action.

1 Introduction

In recent years there has been a great interest in noncommutative field theories. This is due not only because of their features, which constitute remarkable generalizations of those presented by conventional field theories, but also because they naturally appear in the context of string theories [1]. To construct the noncommutative version of a field theory one basically replace the product of fields in the action by the Moyal product:

\[
\phi_1(x) \star \phi_2(x) = \exp \left( \frac{i}{2} \theta^{\mu \nu} \partial_\mu x \partial_\nu y \right) \phi_1(x) \phi_2(y) \big|_{x=y}
\]  

where \( \theta^{\mu \nu} \) is a real and antisymmetric constant matrix.

Regardless the enormous amount of works written on the subject, only a few of them concern the Hamiltonian treatment of noncommutative theories. They consider both the cases depending if the noncommutative parameter \( \theta^{0i} \) vanishes [2, 3] or not [4, 5]. The examples found in literature consider only the Hamiltonian formulation of noncommutative \( U(1) \) gauge theories. In a more general setting, however, formally the Lie commutators of the corresponding nonAbelian commutative theory are replaced by Moyal commutators. This modification implies, among other consequences, that \( SU(N) \) can not consistently be the symmetry group of a noncommutative action [1]. This has remarkable consequences, not only

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related with the structure of the correspondence between commutative and noncommutative
gauge theories [1, 6], but also implies severe restrictions over the phenomenology described
by the theory [7, 8].

In the present letter we will consider the Hamiltonian treatment of a general nonAbelian
noncommutative gauge theory adopting the condition $\theta^{ij} = 0$, to keep unitarity [9] and
avoid non canonical means. We discuss the enveloping algebra structure of the theory and its
relation with the invariance under the $U(N)$ symmetry group and implement the Hamiltonian
treatment of the noncommutative gauge theory, displaying constraints, Hamiltonian, their
first class algebra and the gauge invariance of the corresponding first order action.

2 Hamiltonian noncommutative gauge theory

The action which describes the gauge sector of a noncommutative nonAbelian theory can be
written as

$$S = -\frac{1}{2} \operatorname{tr} \int d^4x F_{\mu\nu} F^{\mu\nu}$$

where the curvature tensor is defined by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i (A_\mu \ast A_\nu - A_\nu \ast A_\mu) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu - i [A_\mu, A_\nu]$$

Action $S$ is invariant under the gauge transformations

$$\delta A_\mu = D_\mu \Lambda = \partial_\mu \Lambda - i [A_\mu, \Lambda]$$

since under (3) $\delta F_{\mu\nu} = -i [F_{\mu\nu}, \Lambda]$ and the Moyal product is associative and satisfies the
cyclic property under the integral sign. It is also easy to deduce that transformations (3)
close in an algebra

$$[\delta_1, \delta_2] A_\mu = \delta_3 A_\mu$$

where $A_3 = i [A_1, A_2]$. However, the gauge transformations (3) imply that the gauge fields
cannot take values, for instance, in a $su(N)$ algebra, but in some section of the corresponding
enveloping algebra [6]. Now, form (3), it follows that

$$\delta A_\mu = \partial_\mu \Lambda^a T^a - i \frac{1}{2} [A_\mu^a, \Lambda^b] \{T^a, T^b\} - i \frac{1}{2} [A_\mu^a, \Lambda^b] [T^a, T^b]$$

and so $A_\mu$ is forced to take values not only along $T^a$ but also along $\{T^a, T^b\}$. When the $T^a$’s
are the generators of some Lie algebra $g$ in some representation $R$, we call this section of the
corresponding enveloping algebra as $u(g, R)$, where the (anti)commutators are constructed
with usual matrix products. The generators of $u(g, R)$ can be written as

$$T^A = (T^a, \frac{1}{2} \{T^a, T^b\}, \frac{1}{4} \{T^a, \{T^b, T^c\}\}, ...)$$
where the range of the index $A$ depends on $u(g, R)$. In general, it is easy to verify that the generators of $u(g, R)$ given by (6) not only form a Lie algebra but also close under anticommutation:

$$[T^A, T^B] = if^{ABC}T^C$$

and

$$\{T^A, T^B\} = d^{ABC}T^C$$

where $f^{ABC} = -f^{BAC}$ and $d^{ABC} = d^{BAC}$. The simpler nontrivial algebra that matches these conditions is $u(N)$ in the representation given by $N \times N$ hermitian matrices. Following [7], one can choose $T^0 = \frac{1}{\sqrt{2N}}1_{N \times N}$ and the remaining $N^2 - 1$ of the $T$’s as in $su(N)$. It is then possible to use the trace condition

$$tr(T^AT^B) = \frac{1}{2}\delta^{AB}$$

and take $f^{ABC}$ and $d^{ABC}$ as completely antisymmetric and completely symmetric respectively. From now on we will assume these conditions. Now we can explicitly write the gauge connection and the curvature as

$$A_\mu = A^A_\mu T^A$$

and

$$F_{\mu\nu} = F^A_{\mu\nu} T^A$$

and in terms of components, (2), (7) and (9) permit us to write, for instance, that

$$F_{\mu\nu} = \left(\partial_\mu A_\nu^D - \partial_\nu A_\mu^D + \frac{1}{2}f^{BCD}(A_\mu^B, A_\nu^C) - i\frac{1}{2}d^{BCD}[A_\mu^B, A_\nu^C]\right)T^D$$

In a similar way, when we expand the gauge parameter along the generators of $u(g, R)$, we observe that (3) and $\delta F_{\mu\nu} = -i[F_{\mu\nu}, \Lambda]$ can be rewritten as

$$\delta A_\mu = \left(\partial_\mu \Lambda^D + \frac{1}{2}f^{BCD}(A_\mu^B, \Lambda^C) - \frac{i}{2}d^{BCD}[A_\mu^B, \Lambda^C]\right)T^D,$$

$$\delta F_{\mu\nu} = \left(\frac{1}{2}f^{BCD}(F_{\mu\nu}^B, \Lambda^C) - \frac{i}{2}d^{BCD}[F_{\mu\nu}^B, \Lambda^C]\right)T^D.$$  

From (8) and (10) we can write the action $S$ as

$$S = -\frac{1}{4} \int d^4x F_{\mu\nu}^A F^{\mu\nu A}.$$  

We are considering the noncommutative parameter to satisfy $\theta_{\mu\nu}\theta^{\mu\nu} > 0$ and assuming a Lorentz referential where $\theta^{0i}$ vanishes identically, we can treat action (12) in a canonical way. So we derive the momenta conjugate to $A^B_\mu$ as

$$\Pi^B_\mu = \frac{\partial L}{\partial A_\mu^B} = F^B_{\mu0}.$$  

This means that there are primary constraints
\[ T^A_1 = \Pi^A_0 \] (14)

and making use of a partial integration as well as of the symmetry properties of the structure functions \( f^{ABC} \) and \( d^{ABC} \), we arrive at the primary Hamiltonian

\[
H_p = \int d^3x \left( \frac{1}{2} \Pi^B \Pi^B + \frac{1}{4} F_{ij}^B F^{ij}_B - (D_i \Pi^i)^B A^0B + \lambda^B T_1^B \right)
\] (15)

where \((D_i \Pi^i)^B = \partial_i \Pi^B + \frac{1}{2} f^{BCD} [A^C, \Pi^D] - \frac{i}{2} d^{BCD} [A^C, \Pi^D] \).

By using the Poisson brackets definition

\[
\{ X(x), Y(y) \}_{PB} = \int d^3z \left( \frac{\delta X(x)}{\delta A^C_\mu (z)} \frac{\delta Y(y)}{\delta \Pi^{C\mu} (z)} - \frac{\delta Y(y)}{\delta A^C_\mu (z)} \frac{\delta X(x)}{\delta \Pi^{C\mu} (z)} \right)
\] (16)

where \( x^0 = y^0 = z^0 \), it is easy to see that there are secondary constraints

\[
\{ T^A_1, H_p \}_{PB} = (D_i \Pi^i)^A \equiv T_2^A
\] (17)

One can show that collectively the constraints satisfy a first class algebra

\[
\{ T^A_1(x), T^B_1(y) \}_{PB} = 0 \text{ and } \{ T^A_1(x), T^B_2(y) \}_{PB} = 0,
\{ T^A_2(x), T^B_2(y) \}_{PB} = \frac{1}{2} f^{ABC} \{ \delta(x - y), T^C_2(x) \} - \frac{i}{2} d^{ABC} [\delta(x - y), T^C_2(x)]
\] (18)

As can be observed, the above expressions present the correct symmetry properties under the change \((xA) \leftrightarrow (yB)\). In a similar way, one can also prove that

\[
\{ T^A_2, H \}_{PB} = \frac{1}{2} f^{ABC} \{ \lambda^{2B} - A^{0B}, T^C_2 \} - \frac{i}{2} d^{ABC} [\lambda^{2B} - A^{0B}, T^C_2] = -i \lambda^2 - A^0B, T^A_2 \]
\] (19)

and so no more constraints are produced. In the above equation, \( H = H_p + \int d^3x \lambda^{2B} T^A_2 \) is the first class Hamiltonian.

Before concluding, let us consider the gauge invariance of the first order action

\[
S_{fo} = \text{tr} \int d^4x \left( 2 \Pi^\mu \dot{A}_\mu - \Pi^i \Pi^i - \frac{1}{2} F_{ij} F^{ij} - 2T_2 (\lambda^2 - A^{0B} - 2T_1 \lambda^1) \right)
\] (20)

The gauge generator \( G = \int d^3x (\epsilon^1 A^T_1 + \epsilon^2 A^T_2) \) acts canonically on the phase space variables \( y \) through \( \delta y = \{ y, G \}_{PB} \) to produce the gauge transformations

\[
\delta A^{0B} = \epsilon^{1B} \text{ and } \delta \Pi^{0B} = 0,
\delta A^B_i = -\langle D_i \epsilon^2 \rangle^B \text{ and } \delta \Pi^i = -i[\epsilon^2, \Pi]^B.
\] (21)
It is now a simple algebraic task to show that indeed (20) is invariant under (21) once

$$\delta \lambda^1 = \dot{\epsilon}^1 \quad \text{and} \quad \delta \lambda^2 = \epsilon^1 + \dot{\epsilon}^2 - i[\lambda^2 - A^{0B}, \epsilon^2]$$

(22)

We conclude this letter by remarking that we have derived a consistent Hamiltonian formulation for the gauge sector of a nonAbelian noncommutative gauge theory, succeeding in displaying the first class constraints and Hamiltonian, their non trivial algebra and the way they generate the evolution and gauge invariance of phase space quantities. Of course it remains to consider not only the inclusion of matter fields but several fundamental points related with the quantization procedure, as the extension of the phase space in order to construct a \textit{BRST} invariant action and its corresponding path integral, with consistent gauge fixing and measure.

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\textbf{References}


