Vortex Solutions in noncommutative space

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Abstract

We present results on non-trivial classical solutions for fiedl theories defined on non-commutative space. In particular we discuss vortex like solutions to trhe self-dual equations of the 2+1 dimensional Maxwell-Higgs model and for the relativistic and non-relativistic Chern-Simnos-Maxwell-Higgs systems.

1 Introduction

The recent interest aroused by quantum field theories in noncommutative space [1]-[3] prompted the search of localized classical solutions in noncommutative geometry. Instantons, solitons carrying various kinds of fluxes, BPS and non-BPS solutions to different noncommutative theories have been presented in refs.[4]-[19]. Among these models, the Abelian-Higgs model in non-commutative space has received particular attention in connection with vortex like solutions [7],[16], [17], [18].

Several vortex solutions that have been discussed up to now are regular at finite noncommutative parameter θ, but they become singular in the limit $\theta \to 0$ [7],[14]. More precisely, the magnetic field B associated to the flux tube behaves as $B \to \delta^{(2)}(x)$ as $\theta \to 0$. This fact is not surprising since these solutions are obtained through a procedure which is the analogous to performing singular gauge transformations leading to topologically non-trivial solutions from trivial ones in commutative space.

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A different class of vortices in non commutative space has been considered by D.P.Jatkar, G.Mandal and S.Wadia [8], which are closer in spirit to the regular Nielsen-Olesen vortices of the theory in ordinary space. More specifically, in [8], *self-dual* Bogomol'nyi equations were derived and the solutions in the limiting cases $\theta \to 0$ (where they are regular) and $\theta \to \infty$ were considered.

We discuss here the results already presented in [17] [18] leading to regular vortex solutions in noncommutative space for the $2+1$ the Abelian Maxwell-Higgs and Abelian Chern-Simons-Higgs models. We solve the associated self-dual equations finding exact (numerical) solutions.

2 Noncommutative space

We consider space-time with coordinates X^μ ($\mu = 0, 1, 2, 3$) obeying the following noncommutative relations

$$[X^\mu, X^\nu] = i\theta^{\mu\nu} \tag{1}$$

We take $\theta^{0i} = 0$ ($i = 1, 2, 3$). Concerning θ^{ij}, it can be can be brought into its canonical (Darboux) form by an appropriate orthogonal rotation

$$[X^1, X^2] = i\theta\,, \qquad [X^1, X^3] = [X^2, X^3] = 0 \tag{2}$$

One way to describe field theories in noncommutative space is by introducing a Moyal product $*$ between ordinary functions. To this end, one can establish a one to one correspondence between operators $\hat{f}$ and ordinary functions f through a Weyl ordering

$$\hat{f}(X^1, X^2) = \frac{1}{2\pi}\int d^2k \tilde{f}(k_1, k_2) \exp\left(i(k_1X^1 + k_2X^2)\right) \tag{3}$$

Then, the product of two Weyl ordered operators $\hat{f}\hat{g}$ corresponds to a function $f * g(x)$ defined as

$$f * g(x) = \exp\left(\frac{i\theta}{2}(\partial_{x_1}\partial_{y_2} - \partial_{x_2}\partial_{y_1})\right) f(x_1, x_2)g(y_1, y_2)\Big|_{x_1=x_2, y_1=y_2} \tag{4}$$

Given a $U(1)$ gauge field $A_\mu(x)$, the field strength $F_{\mu\nu}$ is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i(A_\mu * A_\nu - A_\nu * A_\mu) \tag{5}$$

We shall couple the gauge field to a complex scalar field ϕ with covariant derivative

$$D_\mu\phi = \partial_\mu\phi - iA_\mu * \phi \tag{6}$$

The alternative approach to noncommutative field theories is to directly work with operators in the phase space (X^1, X^2), with commutator (2). In this case the $*$ product is just the product of operators and integration over the (X^1, X^2) plane is a trace,

$$\int dx^1 dx^2 f(x^1, x^2) = 2\pi\theta \mathrm{Tr} \hat{f}(X^1, X^2) \tag{7}$$

In this framework, we introduce complex variables z and $\bar{z}$

$$z = \frac{1}{\sqrt{2}}(x^1 + ix^2)\,, \qquad \bar{z} = \frac{1}{\sqrt{2}}(x^1 - ix^2) \tag{8}$$

and annihilation and creation operators $\hat{a}$ and $\hat{a}^\dagger$ in the form

$$\hat{a} = \frac{1}{\sqrt{2\theta}}(X^1 + iX^2)\,, \qquad \hat{a}^\dagger = \frac{1}{\sqrt{2\theta}}(X^1 - iX^2) \tag{9}$$

so that (2) becomes

$$[\hat{a}, \hat{a}^\dagger] = 1 \tag{10}$$

With this conventions, derivatives are given by

$$\partial_z = -\frac{1}{\sqrt{\theta}}[\hat{a}^\dagger,\ ]\,, \qquad \partial_{\bar{z}} = \frac{1}{\sqrt{\theta}}[\hat{a},\ ] \tag{11}$$

The field strength takes then the form

$$\hat{F}_{z\bar{z}} = \partial_z \hat{A}_{\bar{z}} - \partial_{\bar{z}} \hat{A}_z - i[\hat{A}_z, \hat{A}_{\bar{z}}] = -\frac{1}{\sqrt{\theta}}\left([\hat{a}^\dagger, \hat{A}_z] + [\hat{a}, A_{\bar{z}}] + i\sqrt{\theta}[\hat{A}_z, \hat{A}_{\bar{z}}]\right) \equiv i\hat{B} \tag{12}$$

with $\hat{B}$ the magnetic field. Concerning covariant derivatives

$$\begin{aligned} D_{\bar{z}}\hat{\phi} &= \partial_{\bar{z}}\hat{\phi} - iA_{\bar{z}}\hat{\phi} = \frac{1}{\sqrt{\theta}}[\hat{a}, \hat{\phi}] - i\hat{A}_{\bar{z}}\hat{\phi} \\ D_z\hat{\bar{\phi}} &= \partial_z\hat{\bar{\phi}} + i\hat{A}_z\hat{\bar{\phi}} = -\frac{1}{\sqrt{\theta}}[\hat{a}^\dagger, \hat{\bar{\phi}}] + i\hat{A}_z\hat{\bar{\phi}} \end{aligned} \tag{13}$$

where

$$\hat{A}_z = \frac{1}{\sqrt{2}}(\hat{A}_1 - i\hat{A}_2)\,, \quad \hat{A}_{\bar{z}} = \frac{1}{\sqrt{2}}(\hat{A}_1 + i\hat{A}_2) \tag{14}$$

3 The $2+1$ Maxwell Higgs model

Dynamics for the model will be governed by the Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu} * F^{\mu\nu} + \overline{D_\mu\phi} * D^\mu\phi - \frac{1}{2}(\phi * \bar{\phi} - \eta^2)^2] \tag{15}$$

Here we have chosen coefficient of the symmetry breaking potential at the Bogomol'ny point [21]-[22]. We are looking for static axially symmetric Nielsen-Olesen vortices with $A_0 = A_3 = 0$. Then, the only relevant coordinates in the problem will be $i = 1, 2$.

The energy functional associated to action (40) can be then written as [8]

$$E = 2\pi\theta \mathrm{Tr}\left(\frac{1}{2}\hat{B}^2 + D_{\bar{z}}\hat{\phi}D_z\hat{\bar{\phi}} + D_z\hat{\phi}D_{\bar{z}}\hat{\bar{\phi}} + \frac{1}{2}(\hat{\phi}\hat{\bar{\phi}} - \eta^2)^2\right) \quad (16)$$

We want to find static solutions minimizing the energy. To this end, we shall proceed à la Bogomol'nyi writing the energy E in the two following forms [8]

$$E = 2\pi\theta \mathrm{Tr}\left(\frac{1}{2}\left(\hat{B} + (\hat{\phi}\hat{\bar{\phi}} - \eta^2)\right)^2 + 2D_{\bar{z}}\hat{\phi}D_z\hat{\bar{\phi}} + (\hat{T}^s + \eta^2\hat{B})\right) \quad (17)$$

with $\hat{T}^s$ defined as

$$\hat{T}^s = \partial_z((D_{\bar{z}}\hat{\phi})\hat{\bar{\phi}}) - \partial_{\bar{z}}((D_z\hat{\phi})\hat{\bar{\phi}}) \quad (18)$$

or

$$E = 2\pi\theta \mathrm{Tr}\left((\frac{1}{2}\left(\hat{B} - (\hat{\phi}\hat{\bar{\phi}} - \eta^2)\right)^2 + 2D_{\bar{z}}\hat{\bar{\phi}}D_z\hat{\phi} - (\hat{T}^a + \eta^2\hat{B})\right) \quad (19)$$

with

$$\hat{T}^a = -\hat{T}^s \quad (20)$$

Now, one can easily see that $\mathrm{Tr}\hat{T}^a = 0$ [8] and hence the energy is bounded by the magnetic flux, as in the case of vortices in ordinary space. The bound is attained when the following first order Bogomol'nyi eqs. hold

$$\hat{B} = \eta^2 - \hat{\phi}\hat{\bar{\phi}}, \qquad D_{\bar{z}}\hat{\phi} = 0 \qquad \text{self−dual equations} \quad (21)$$

or

$$-\hat{B} = \eta^2 - \hat{\phi}\hat{\bar{\phi}}, \qquad D_z\hat{\phi} = 0 \qquad \text{anti self−dual equations} \quad (22)$$

We have fixed in eqs.(21)-(22) our terminology. Eqs.(21) are called *self-dual equations* while eqs.(22) are the corresponding *anti self-dual equations*. Solutions to eqs.(21) correspond to positive magnetic flux, while those to (22) give negative magnetic flux. Note that our convention coincide with that in ([8]) and is the opposite to that in [16], where, in our terminology, anti self-dual solutions are discussed in detail and a critical value of the noncommutative parameter is found, $\theta_c = 1/\eta^2$, such that solutions cease to exist when $\theta > \theta_c$. Now, as stressed above, in the noncommutative case, the presence of the parity breaking θ parameter renders the connection between the anti self-dual and the self-dual case non-trivial, in contrast to what happens in the commutative case where it is straightforward.

In what follows, we construct *exact solutions to the self-dual equations* (21) for arbitrary values of θ and in this sense, our calculation complements those in [8] and [16]. To this end, we propose the following ansatz

$$\hat{A}_z = \frac{i}{\sqrt{\theta}}\sum_n(\sqrt{n+1} - \sqrt{n+2} + e_n)|n+1\rangle\langle n| \quad (23)$$

$$\hat{\phi} = \eta\sum_n f_n|n\rangle\langle n+1| \quad (24)$$

Notice that the Higgs field can be rewritten as

$$\hat{\phi} = \eta \frac{f(\hat{N})}{\sqrt{\hat{N}+1}} \frac{X^1 + iX^2}{\sqrt{2\theta}} \tag{25}$$

where $\hat{N} = \hat{a}^\dagger \hat{a}$ and $\langle | f(\hat{N}) | n \rangle = f_n$. This sould be compared with the ansatz in the commutative case,

$$\phi = \eta\, g(|z|)\, z \tag{26}$$

with $g(0)$ to be determined by solving the Bogomol'nyi equations and requiring that at infinity $g(|z|) \sim 1/|z|$. This has been done in [21] with the result

$$g(0)^2 = 0.72791 \tag{27}$$

In the same way, introducing the ansatz (23)-(24) we expect to derive a recurrence relation for f_n whose solution is uniquely determined by requiring that $f(\infty) \to 1$.

Notice also that the flux-tube solutions presented in [6],[14] correspond to the choice of coefficients $e_n = 0$ and $f_n = 1$, leading to "quasi pure gauge" solutions (which, in the $\theta \to 0$ limit give singular vortex solutions with magnetic field $B = \delta^{(2)}(x)$). What we are looking for here is to determine, through recurrence relations deriving from (21)-(24), the non-trivial values for e_n, f_n that correspond to exact solutions, which should lead to the regular ones found in [21] in the commutative $\theta \to 0$ case. In fact, this ansatz can be seen as the analogous to performing, in the commutative case, a $U(1)$ singular gauge transformation $\exp(in\varphi)$ on $|\phi(r)|$; the condition $|\phi(0)| = 0$ ensures the regularity of the solution. In noncommutative space, the equivalent of such a procedure is to apply an operator S^n with $\hat{S}$ the shift operator defined as [6]

$$\hat{S} = \sum_k |k\rangle\langle k+1| \tag{28}$$

Ansatz (24) just corresponds to a combination of bra and kets like in S but with arbitrary coefficients f_n. It is easy to also see that the compatible ansatz for the gauge field is just (23).

Now, in order to determine the up to now arbitrary coefficients f_n, e_n, we plug ansatz (23)-(24) in eqs.(21) getting the following recurrence relations

$$\begin{aligned} \sqrt{(n+2)}(f_{n+1} - f_n) - e_n f_{n+1} &= 0 \\ 2\sqrt{(n+1)}e_{n-1} - e_{n-1}^2 - 2\sqrt{(n+2)}e_n + e_n^2 &= -\theta\eta^2(f_n^2 - 1) \end{aligned} \tag{29}$$

This coupled system can be combined to give for f_n

$$\begin{aligned} f_1^2 &= \frac{2f_0^2}{1 + \theta\eta^2 - \theta\eta^2(f_0^2)} \\ f_{n+1}^2 &= \frac{(n+2)f_n^4}{f_n^2 - \theta\eta^2 f_n^2(f_n^2 - 1) + (n+1)f_{n-1}^2} \qquad n > 0 \end{aligned} \tag{30}$$

Given a value for f_0 one can then determine all f_n's from (30). The correct value for f_0 should make $f_n^2 \to 1$ asymptotically so that boundary conditions are satisfied. The values of these coefficients will depend on the choice of the dimensionless parameter $\theta\eta^2$.

For small θ we have checked that we re-obtain the values for the commutative solution. Indeed,

$$\frac{f_0^2}{2\eta^2\theta} = 0.72792 \qquad \theta << 1 \tag{31}$$

(compare with eq.(27)), while for large θ we reobtain the result of ref. [8]

$$f_0^2 = 1 - \frac{1}{\eta^2\theta} \qquad \theta >> 1 \tag{32}$$

Exploring the whole range of $\theta\eta^2$, one finds that the vortex solution with +1 units of magnetic flux exists in all the intermediate range. As an example, we list three representative values,

$$\begin{aligned} \theta\eta^2 &= 0.5, \qquad & f_0^2 &= 0.40069\ldots \\ \theta\eta^2 &= 1.0, \qquad & f_0^2 &= 0.56029\ldots \\ \theta\eta^2 &= 2.0, \qquad & f_0^2 &= 0.70670\ldots \end{aligned} \tag{33}$$

Once all $f_n's$ and $e_n's$ are calculated, one can compute the magnetic field, using for example the formula

$$\hat{B} = \eta^2 \sum_{n=0}^{\infty} \left(1 - f_n^2\right) |n\rangle\langle n| \tag{34}$$

or, using the explicit formula for $|n\rangle\langle n|$ in configuration space [6]

$$B(r) = 2\eta^2 \sum_{n=0}^{\infty} (-1)^n \left(1 - f_n^2\right) \exp(-\frac{r^2}{\theta}) L_n(2\frac{r^2}{\theta}) \tag{35}$$

where L_n are the Laguerre polynomials.

We show in figure 1 the resulting magnetic field B as a function of $r\eta$. For $\theta = 0$ one recovers the result for self dual Nielsen-Olesen vortices in ordinary space [21]. As θ grows, the maximum for B decreases and the vortex is less localized with total area such that the magnetic flux remains equal to 1. It is important to stress that we have found noncommutative self-dual vortex solutions in the whole range of θ, in agreement with the analysis for large and small θ presented in [8].

As θ becomes larger, one needs more and more precision in order to match the value of f_0 so that the vortex has the adequate behavior at infinity, but a solution can be always found (this should be contrasted with the anti self-dual case discussed in [16]). One can easily integrate $B(r)$ in (35) and check that the magnetic flux Φ, which can also be written as

$$\Phi = 2\pi\theta \mathrm{Tr}\hat{B} \tag{36}$$

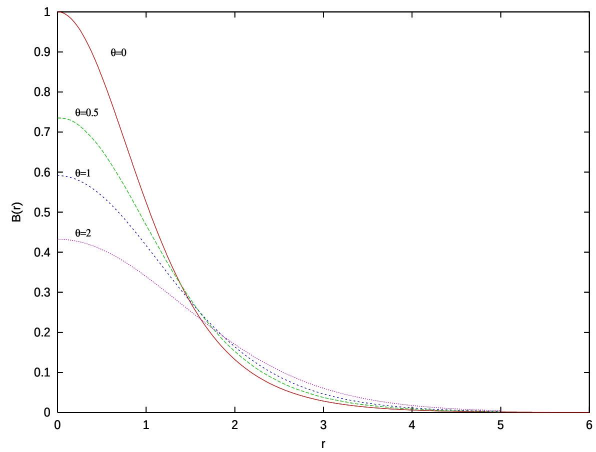


Figure 1: Magnetic field of the vortex as a function of the radial coordinate (in units of η) for different values of the anticommuting parameter θ (in units of η^2). The curve for $\theta = 0$ coincides with that of the ordinary Nielsen-Olesen vortex.

gives, for the exact solution,

$$\frac{\Phi}{2\pi} = 1 \tag{37}$$

We have also computed the energy by inserting our vortex solution directly in eq.(17). As expected, the solution saturates the bound giving

$$E = 2\pi\eta^2 \tag{38}$$

4 The $2+1$ Chern-Simons Higgs model

In ordinary $2+1$ dimensional space, models of relativistic and non-relativistic matter minimally coupled to gauge fields whose dynamics is governed by a CS term have self-dual vortex-like solutions[23]-[25].

We will be interested in the noncommutative extension of the nonrelativistic and relativistic Chern-Simons-matter systems introduced, in ordinary space, in refs.[23]-[25]. The gauge field dynamics for these models is governed by the Chern-Simons Lagrangian $L_{CS}[A]$ defined as

$$L_{CS}[A] = \kappa\, \varepsilon_{\mu\nu\alpha} \left(A_\mu * \partial_\nu A_\alpha - \frac{2i}{3} A_\mu * A_\nu * A_\alpha) \right) \tag{39}$$

The Lagrangian for the noncommutative extension of the non-relativistic case will be taken as

$$L = L_{CS}[A] + i\bar\phi * D_0\phi + \frac{1}{2}\overline{D_i\phi} * D_i\phi - \frac{1}{4}\lambda\phi * \bar\phi * \phi * \bar\phi \tag{40}$$

while for the relativistic case,

$$L = L_{CS}[A] + \overline{D_\mu \phi} * D^\mu \phi - V[\phi * \bar{\phi}] \tag{41}$$

with V the sixth order potential

$$V[\phi * \bar{\phi}] = \frac{1}{\kappa^2} \phi * \bar{\phi} * \left(\phi * \bar{\phi} - v^2\right)^2 \tag{42}$$

taken at the selfdual point, where Bogomol'nyi equations can be found.

4.1 BPS equations for the non-relativistic case

The Hamiltonian associated with Lagrangian (40) is simply given by

$$H = \int d^2x \left(\frac{1}{2} \overline{D_i \phi} * D_i \phi + \frac{1}{4} \lambda \phi * \bar{\phi} * \phi * \bar{\phi} \right) \tag{43}$$

It can be written in the form

$$H = \int d^2x \left(-\frac{1}{2} \bar{\phi}(D_1 + i\alpha D_2)(D_1 - i\alpha D_2)\phi + \phi * \bar{\phi} * \left(-\frac{\alpha}{2} F_{12} + \frac{1}{4} \lambda \phi * \bar{\phi} \right) \right) \tag{44}$$

where $\alpha = \pm 1$. We shall call $\alpha = -1$ the *selfdual* case and $\alpha = +1$ the *anti selfdual* one.

Using the Gauss law deriving from Lagrangian (40),

$$\kappa \varepsilon_{ij} F_{ij} + \phi * \bar{\phi} = 0 \tag{45}$$

the Hamiltonian takes the form

$$H = \int d^2x \left(-\frac{1}{2} \bar{\phi}(D_1 + i\alpha D_2)(D_1 - i\alpha D_2)\phi - \frac{1}{2}(\alpha + \lambda\kappa)\phi * \bar{\phi} * F_{12} \right) \tag{46}$$

Then, if the following relation among the two free parameters in the theory holds

$$\lambda \kappa = -\alpha \tag{47}$$

the lower bound for the Hamiltonian is attained when the following Bogomol'nyi equations are satisfied

$$\begin{aligned} (D_1 - i\alpha D_2)\, \phi &= 0 \\ B &= -\frac{1}{2\kappa} \phi * \bar{\phi} \end{aligned} \tag{48}$$

Let us first consider the selfdual ($\alpha = -1$) case. In operator language, equations (48) can then be written as

$$\begin{aligned} D_{\bar{z}} \hat{\phi} &= 0 \\ B &= -\frac{1}{2\kappa} \hat{\phi} \hat{\bar{\phi}} \end{aligned} \tag{49}$$

In order to search for vortex solutions to these equations, we propose the ansatz

$$\hat{\phi} = \sqrt{\frac{2|\kappa|}{\theta}} \sum_{n=0}^{\infty} f_n |n\rangle\langle n+M-1| \tag{50}$$

$$\hat{A}_z = \frac{i}{\sqrt{\theta}} \sum_{n=0}^{\infty} d_n |n+1\rangle\langle n| \tag{51}$$

where f_n and d_n are arbitrary real coefficients and $\{|n\rangle\}$ is the basis provided by the number operator $\hat{N}$. The ansatz (50) leads, in the $\theta \to 0$ limit, to $\phi \sim \rho(r) z^{M-1}$ which corresponds, in ordinary space, to the usual cylindrically symmetric ansatz with a Higgs field phase $(M-1)\varphi$ [23].

Inserting ansatz (50)-(51) into eq.(49) one obtains the following recurrence relations

$$\begin{aligned} &2\sqrt{p}\, d_{p-1} - d_{p-1}^2 - 2\sqrt{p+1}\, d_p + d_p^2 = -\frac{|\kappa|}{\kappa} f_p^2 \\ &d_p = \sqrt{p+1} - \sqrt{p+M}\frac{f_p}{f_{p+1}} \end{aligned} \tag{52}$$

which can be combined into the following recurrence relation for the $f_n's$ coefficients

$$\begin{aligned} f_1^2 &= \frac{M f_0^2}{1 - \frac{|\kappa|}{\kappa} f_0^2} \\ f_{p+1}^2 &= \frac{(p+M) f_p^2}{1 - \frac{|\kappa|}{\kappa} f_p^2 + (p+M-1) f_{p-1}^2 / f_p^2}\,, \quad p \geq 1 \end{aligned} \tag{53}$$

If, instead, we choose the anti selfdual case ($\alpha = 1$), the equations to solve read

$$\begin{aligned} &D_z \hat{\phi} = 0 \\ &B = -\frac{1}{2\kappa} \hat{\phi} \hat{\bar{\phi}} \end{aligned} \tag{54}$$

In this case, the appropriate ansatz is

$$\begin{aligned} \hat{\phi} &= \sqrt{\frac{2|\kappa|}{\theta}} \sum_{n=0}^{\infty} f_n |n+M-1\rangle\langle n| \\ \hat{A}_z &= \frac{i}{\sqrt{\theta}} \sum_{n=0}^{\infty} d_n |n+1\rangle\langle n| \end{aligned} \tag{55}$$

and the recurrence relations become

$$\begin{aligned} &2\sqrt{p}\, d_{p-1} - d_{p-1}^2 - 2\sqrt{p+1}\, d_p + d_p^2 = -\frac{|\kappa|}{\kappa} f_{p-1}^2 \\ &d_{p+1} = \sqrt{p+M} - \sqrt{p+1}\frac{f_{p+1}}{f_p} \end{aligned} \tag{56}$$

combining to

$$
\begin{aligned}
f_1^2 &= \left(M - \frac{|\kappa|}{\kappa} f_0^2\right) f_0^2 \\
f_{p+1}^2 &= \frac{1}{p+1} f_p^2 \left(-\frac{|\kappa|}{\kappa} f_p^2 + 1 + p\frac{f_p^2}{f_{p-1}^2}\right), \quad p \geq 1
\end{aligned}
\tag{57}
$$

The flux of the solutions is given by

$$
\frac{\Phi}{2\pi} = \theta \mathrm{Tr} B = -\frac{|\kappa|}{\kappa} \sum_p f_p^2 \tag{58}
$$

Finite flux configurations correspond to solutions such that

$$
\lim_{p\to\infty} f_p = 0 \tag{59}
$$

The analysis of the asymptotic behavior of the recurrence relation (52) shows that, for large p,

$$
f_p^2 \to \frac{1}{p^\beta} \tag{60}
$$

with β a real positive parameter to be determined. The flux can also be obtained directly from the expression of B as

$$
\frac{\Phi}{2\pi} = \lim_{p\to\infty} \left(d_p^2 - 2d_p\sqrt{p+1}\right) \tag{61}
$$

Using eqs. (52)-(56) one gets,

$$
\frac{\Phi}{2\pi} = M - 1 + \beta \qquad \kappa < 0 \tag{62}
$$

$$
\frac{\Phi}{2\pi} = -M + 1 - \beta \qquad \kappa > 0 \tag{63}
$$

The numerical study of the recurrence relations reveals that eq.(52) has solutions only for $\kappa < 0$ while eq.(56) has solutions only for $\kappa > 0$. Thus, as in the commutative case, solutions exist only for $\lambda < 0$, that is, only for an "attractive" interaction. Given an initial arbitrary value f_0^2 for the recurrence relation, all coefficients can be determined in such a way that they satisfy (60). Since θ does not enter explicitly in the recurrence relations, one can construct a whole family of solutions parametrized by θ (which appears as a factor, see eqs.(51) or (55)).

We have explored numerically the whole $f_0^2 \geq 0$ range finding, for the selfdual case, that there exists, for every value of f_0, a consistent solution. In contrast, in the anti-selfdual case, the solution ceases to exist for $f_0^2 > M$. This is reminiscent of what happens for non-commutative Nielsen-Olesen anti-selfdual vortices [16].

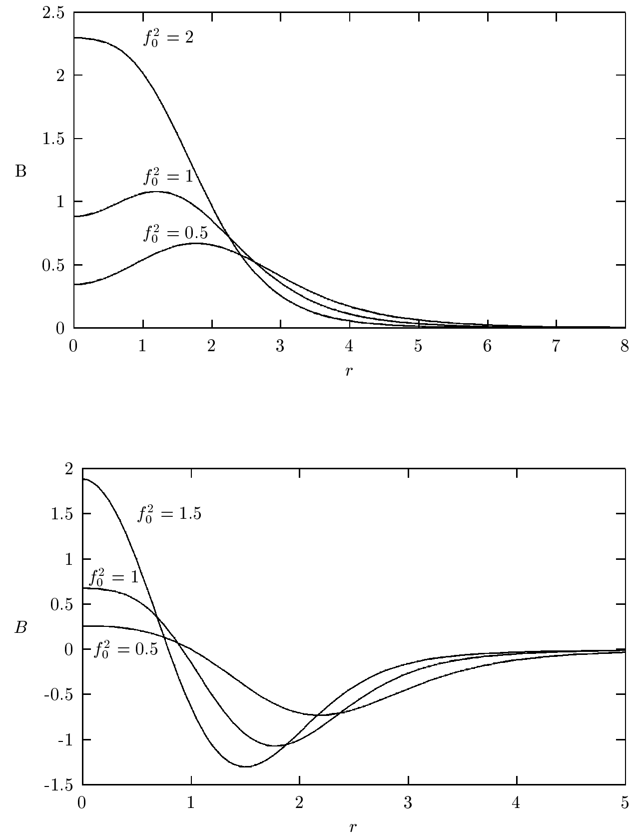


Figure 2: The magnetic field $B(r)$ for the dual and anti-selfdual non-relativistic solution for different values of f_0. θ and κ have been taken equal to 1.

We have find that β is a monotonically increasing function of f_0^2 such that

$$\lim_{f_0 \to 0} \beta = M + 1 \tag{64}$$

Now, according to eqs.(62)-(63) the magnetic flux of our exact solution is in general not quantized in non-commutative space.

We show in figure 2 the magnetic field B as a function of r, for the selfdual and anti-selfdual solutions, computed from eqs.(49),(54). We plot different values of f_0^2 for a given θ. One sees that the B is reminiscent of the magnetic field corresponding to the ordinary (commutative) case, except that in the latter case $B(0) = 0$ while in the present noncommutative case $B(0) = B_0$, with B_0 positive (negative) for the selfdual (anti-selfdual) case.

Let us relate at this point this result with that corresponding to ordinary space. As originally shown in [23] Bogomol'nyi equations for the non-relativistic system can be exactly solved since the problem can be reduced to finding solutions to the Liouville equation. Then, the most general axially symmetric regular solution gives, for the selfdual case,

$$\phi^{comm} = \frac{2\sqrt{2|\kappa|}}{r} M \left(\left(\frac{r_0}{r}\right)^M + \left(\frac{r}{r_0}\right)^M \right)^{-1} \exp\left(i(M-1)\varphi\right) \ , \quad M = 1, 2, \dots \tag{65}$$

where r_0 is an integration constant. The other free parameter is M which is quantized on regularity grounds. Accordingly, the magnetic flux associated to this solution is quantized,

$$\Phi^{comm} = 2\pi(2M) \tag{66}$$

Note that the flux of our selfdual noncommutative solutions coincides, in the $f_0 \to 0$ limit with that of the ordinary case. To study the connection between our solution and that in ordinary space in more detail, let us consider the $\theta \to 0$ limit of the former in configuration space. It is enough to consider the small r region where the commutative solution (65) can be written in the form

$$\phi^{comm} = \frac{2\sqrt{2|\kappa|}}{r_0^M} M z^{M-1} + O(r^{3M-1}) \tag{67}$$

Since, for small θ, $r^2 \approx \theta\hat{N}$, the leading contribution in the noncommutative case corresponds to the $n = 0$ term in solution (50),

$$\begin{aligned} \hat{\phi} &\approx \sqrt{\frac{2|\kappa|}{\theta}} f_0 |0\rangle\langle M-1| = \sqrt{\frac{2|\kappa|}{\theta}} f_0 |0\rangle\langle 0| \frac{a^{M-1}}{\sqrt{(M-1)!}} \\ &\approx \sqrt{\frac{2|\kappa|}{\theta}} f_0 |0\rangle\langle 0| \frac{z^{M-1}}{\sqrt{(2\theta)^{M-1}(M-1)!}} \end{aligned} \tag{68}$$

A relation between r_0 and f_0 can be found comparing eqs.(67) and (68)

$$f_0^2 = 2^{M+1} M! M \left(\frac{\theta}{r_0^2}\right)^M \tag{69}$$

If f_0^2 does not vanish as θ^M in the $\theta \to 0$ limit, the noncommutative solutions goes in this limit to a singular solution in ordinary space. Only when the behavior (69) is satisfied, the $\theta \to 0$ limit converges to the Jackiw-Pi solution [23]. We have numerically checked this finding that, already for $\theta/r_0^2 \approx 0.01$, the noncommutative and the Jackiw-Pi solutions are indistinguishable.

4.2 BPS equations for the relativistic model

The associated Hamiltonian for the model (41) for static field configurations is

$$H = \int d^2x \left(\overline{D_i\phi} * D_i\phi + A_0 * A_0 * \phi * \bar{\phi} + V[\phi * \bar{\phi}] \right) \tag{70}$$

The Gauss law deriving from (41) takes, for static configurations, the form

$$2\kappa B = -\left(\phi * \bar{\phi} * A_0 + A_0 * \phi * \bar{\phi} \right) \tag{71}$$

Assuming that A_0 (Moyal) commutes with $\phi * \bar{\phi}$ (as it will be the case for our ansatz, see below) we have

$$A_0 = -\kappa(\phi * \bar{\phi})^{-1} * B \tag{72}$$

Inserting (72) in (70) and integrating by parts one gets

$$\begin{aligned} H &= \int d^2x \left(\overline{(D_1 + i\alpha D_2)\phi} * (D_1 - i\alpha D_2)\phi + \right. \\ & \left. \frac{\kappa^2}{|\phi|^2} * \left(B + \alpha \frac{1}{2\kappa^2} |\phi|^2 * (|\phi|^2 - v^2) \right)^2 \right) - \alpha v^2 \Phi \end{aligned} \tag{73}$$

where $\alpha = \pm 1$, $|\phi|^2 = \phi * \bar{\phi}$ and $\Phi = \int d^2x B$ is, as before, the magnetic flux. Thus, in the *selfdual case* ($\alpha = -1$) the energy is bounded by $v^2\Phi$, and the bound is saturated when the selfdual equations are fulfilled

$$\begin{aligned} & D_{\bar{z}}\phi = 0 \ , \\ & B + \frac{1}{2\kappa^2}|\phi|^2 * (|\phi|^2 - v^2) = 0 \end{aligned} \tag{74}$$

Analogously, in the *anti-selfdual* case ($\alpha = 1$) the energy bound is $-v^2\Phi$ and is reached when the anti-selfdual equations are satisfied

$$\begin{aligned} & D_z\phi = 0 \ , \\ & B - \frac{1}{2\kappa^2}|\phi|^2 * (|\phi|^2 - v^2) = 0 \end{aligned} \tag{75}$$

Let us analyze the selfdual case first. As in the non-relativistic case we shall work in the operator framework and propose an ansatz of the form

$$\hat{\phi} = v \sum_{n=0}^{\infty} f_n |n\rangle\langle n + M| \tag{76}$$

$$\hat{A}_z = \frac{i}{\sqrt{\theta}} \sum_{n=0}^{\infty} d_n |n+1\rangle\langle n| \tag{77}$$

where again f_n and d_n are arbitrary real coefficients. With this ansatz both the magnetic field and $|\phi|^2$ are diagonal

$$B = \frac{1}{\theta} \left(\sum_{n=1} (g_{n-1} - g_m) |n\rangle\langle n| - g_0 |0\rangle\langle 0| \right) \tag{78}$$

$$|\hat{\phi}|^2 = v^2 \sum_{n=0} f_n^2 |n\rangle\langle n| \tag{79}$$

where $g_n = 2d_n\sqrt{n+1} - d_n^2$. Consequently, from the Gauss law (71), we see that A_0 commutes with $|\hat{\phi}|^2$ and can be solved as in equation (72).

The selfdual system (74) is then equivalent to the following system of recurrence relations

$$\begin{aligned}
d_p &= \sqrt{p+1} - \sqrt{p+M}\,\frac{f_p}{f_{p+1}}\,, \quad p \geq 0 \\
f_1^2 &= \frac{(M+1)f_0^2}{1 + a\, f_0^2(1-f_0^2)} \\
f_{p+1}^2 &= \frac{(p+M+1)f_p^2}{1 + a\, f_p^2(1-f_0^2) + (p+M)f_{p-1}^2/f_p^2}\,, \quad p \geq 1
\end{aligned} \tag{80}$$

where $a = v^4\theta/(2\kappa^2)$.

For the anti-selfdual case, the ansatz we propose is

$$\hat{\phi} = v \sum_{n=0}^{\infty} f_n |n+M\rangle\langle n| \tag{81}$$

$$\hat{A}_z = \frac{i}{\sqrt{\theta}} \sum_{n=0}^{\infty} d_n |n+1\rangle\langle n| \tag{82}$$

The magnetic field has the same form as in (79) while one has for the scalar field

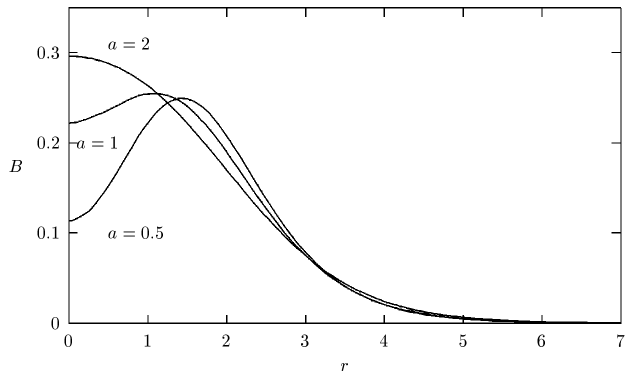
$$|\hat{\phi}|^2 = v^2 \sum_{n=0} f_n^2 |n+M\rangle\langle n+M| \tag{83}$$

The anti-selfdual recurrence relation take the form

$$\begin{aligned}
d_{p+1} &= \sqrt{p+M+1} - \sqrt{p+1}\,\frac{f_{p+1}}{f_p} \\
f_1^2 &= \left(M + 1 - a f_0^2(1-f_0^2)\right) f_0^2 \\
f_{p+1}^2 &= \frac{1}{p+1} f_p^2 \left(1 - a f_p^2(1-f_p^2) + p\frac{f_p^2}{f_{p-1}^2}\right)\,, \quad p \geq 1
\end{aligned} \tag{84}$$

We have studied systems (80) and (84) numerically. Given a value for f_0 one can then determine all f_n's from (80) or (84). The correct value for f_0 should make $f_n^2 \to 1$ asymptotically so that boundary conditions are satisfied (we are looking for symmetry breaking solutions). The values of these coefficients will depend on the choice of the dimensionless parameter $a = \theta v^4/(2\kappa^2)$. We have explored the whole range of a and found a consistent solution for any positive integer M both in the selfdual and in the anti-selfdual case. In contrast with the non-relativistic model, there is only one value of f_0 leading to the appropriate boundary condition, for each value of a. For example, for the self-dual case we have

$$\begin{aligned}
a = 0.5, &\qquad f_0^2 = 0.2168142\ldots \\
a = 1.0, &\qquad f_0^2 = 0.4037747\ldots \\
a = 2.0, &\qquad f_0^2 = 0.6228436\ldots
\end{aligned} \tag{85}$$



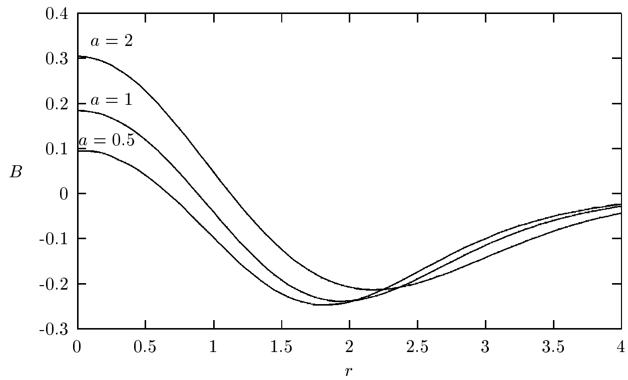


Figure 3: The magnetic field $B(r)$ for the anti-selfdual relativistic solution for different values of $a = v^2\theta/(2\kappa^2)$.

Once the $f_n's$ and $d_n's$ are determined in this way, the magnetic field can be computed using eq.(78) or Bogomol'nyi equation. As an example, for the selfdual case, one has

$$\hat{B} = \frac{v^2}{2\kappa^2}\sum_{n=0}^{\infty} f_n^2\left(1-f_n^2\right)|n\rangle\langle n| \tag{86}$$

or, using the explicit formula for $|n\rangle\langle n|$ in configuration space [6]

$$B(r) = \frac{v^2}{\kappa^2}\sum_{n=0}^{\infty}(-1)^n f_n^2\left(1-f_n^2\right)\exp(-\frac{r^2}{\theta})L_n(2\frac{r^2}{\theta}) \tag{87}$$

where L_n are the Laguerre polynomials.

We show in figure 3 for the selfdual case and anti-selfdual cases the resulting magnetic field B as a function of r for different values of θ. For $\theta = 0$ we recover in both cases the CS vortex solutions found in [24]-[25]. One should note that, as in the non-relativistic case, the magnetic field profile corresponding to the anti-selfdual case is not the trivial reverse of the selfdual one. This is related, as before, to the presence of the parity breaking parameter θ. As θ grows, the magnetic field differs more and more from the annulus-shaped ordinary CS vortex with a value at the origin which grows till $B(0)$ becomes a maximum. It is important to stress that we have found vortex solutions in the whole range of θ *both in the selfdual and anti-selfdual cases* in contrast with what happens for Nielsen-Olesen vortices where anti-selfdual solutions do not exist for θ larger than a critical value [16].

The magnetic flux of the solutions can be computed using

$$\Phi = 2\pi\theta \mathrm{Tr}\hat{B} \tag{88}$$

One finds,

$$\Phi = 2\pi M\,, \quad M = 1, 2, \ldots \tag{89}$$

showing that in the relativistic case, the magnetic flux is quantized for all θ. This, and the expression (73) for the Hamiltonian allows to write the energy of the selfdual ($\alpha = -1$) and anti-selfdual ($\alpha = 1$) solitons in the form

$$H = (2\pi v^2)M \tag{90}$$

which coincides with the expression for CS solitons in ordinary space first found in [24]-[25]

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